Digital Currency Runs

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Abstract

Digital currency created by the private sector, such as bitcoin, is designed to have a determined supply and enable payments with the premise of competing with and supplanting central bank fiat money and the banking system. Central banks are developing fiat central bank digital currency (CBDC) and banks are innovating in response. This paper shows that central bank monetary policy interest rates paid on bank reserves must be set dynamically and relative to interest rates (even if zero rates) paid on CBDC to prevent the disintermediation of banks, support optimal firm investment and risk sharing for consumers, and prevent digital currency runs into CBDC. Private digital currency may be preferred over fiat money in countries with high inflation, but using it outside of the banking system reduces investment and risk sharing. Banks can re-emerge by taking deposits and lending in private digital currency to increase investment and risk-sharing while avoiding fiat inflation, but these banks risk having runs into the private digital currency. Private digital currency is superior to traditional hard currencies, such as based on a gold standard, for investment, risk-sharing and financial stability.

Keywords: Digital currency, CBDC, financial intermediation, bank runs, investment, fiat money, inflation

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1 Introduction

The rapid development of digital currency has renewed traditional questions about whether money creation should be handled by the private sector or public sector. Two original motivations for privately created digital currency such as bitcoin are to act as a replacement for fiat money, which can be inflationary, and banks, which provide payments, as emphasized by Raskin and Yermack (2018). They also highlight that with central banks worldwide developing public digital currency, there are widespread expectations and concerns that private or public digital currency will eventually replace fractional reserve banking.[2] Yet, new forms of banks that take deposits and lend in digital currency are emerging.[3] However, the role of digital currency for supporting lending and investment in the economy without banks has been little studied. Concerns about digital currency creating fragility in the financial system have also been little formulated or studied.[4]

This paper studies the potential competition of privately created digital currency, as well as publicly created central bank digital currency (CBDC), to the traditional roles of fiat money and the banking system. While cryptocurrency has a variety of features, two key elements are considered here. It is typically created with an ultimate fixed supply to avoid inflation, and it provides a means of payment without banks.

The model features consumers with an endowment, firms that need to borrow to invest, and banks that can intermediate payments and loans using fiat money created by the central bank. With optimal fiat inflation, banks can provide maturity and risk transformation to implement optimal investment, economic output, and risk sharing for consumers. However, excessive inflation reduces investment and long-term output.

Private digital currency that is not inflationary can be created to allow for payments and lending by consumers directly to firms without using fiat money or banks, but this lending does not provide for maturity transformation and risk sharing. Banks that take deposits and lend in private digital currency may provide for greater invest-

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[2] For example, see Vigna and Casey (2015).

[3] See Bech and Rodney (2017), Bordo and Levin (2017), Broadbent (2016), Cawrey (2014), Jack and Suri (2011), Popper (2015), and Rosenfeld (2015).

[4] Posner (2015a,b) is one of the few to point out that if bitcoin, for example, were to become widely adopted, fractional reserve banking denominated in bitcoin would be a natural outcome because of the value creation that banks provide, but that financial crises in a bitcoin-based banking system would also occur. See also Winkler (2015) and Nelson (2017).

ment and risk sharing, with private digital currency held as a form of private reserves to enable standard fractional reserve banking. Public digital currency also allows for payments and lending without banks, but again with a lower amount of investment and risk sharing. Public digital currency incurs the same inflation as standard fiat money, and hence consumers prefer to hold it in the form of bank deposits rather than hold or lend it directly.

Banks face the threat of inefficient runs when investment returns are low. Central bank fiat inflation prevents such insolvency-based runs for banks with deposits of fiat money and public digital currency but not private digital currency. Banks with private digital currency deposits have runs into the digital currency precisely because it is not inflationary and can be stored and used for payments outside of the banking system. There is a trade-off for using private digital currency to avoid fiat inflation. If consumers hold it or lend it to firms directly, it does not permit as much credit to firms for investment as banks can provide. If instead it is held in the form of bank deposits, it is subject to fragility in the form of digital currency runs.

However, private digital currency is superior to hard currencies, such as based on a gold standard, that are traditionally used to prevent fiat inflation. A hard currency has a fixed value that makes banks even more susceptible to insolvency-based runs as well as liquidity-based runs. A private digital currency has a partially flexible value that supports greater investment, gives better risk sharing among consumers for macro asset and liquidity risk, and reduces bank runs relative to using a hard currency.

An economy with a monetary system based on a private digital currency instead of central bank fiat money is a viable possibility, as argued by Raskin and Yermack (2018). Bitcoin has been recently adopted as the official legal currency of El Salvador and has had increasing use at times in other countries with high inflation problems including Venezuela, Iran, Argentina, Ukraine, Zimbabwe, and other countries.[5]

Indeed, the Federal Reserve was originally created for the primary purpose of being able to provide an "elastic supply of currency" in order to help banks and the economy weather aggregate liquidity and recessionary risks. But, as with other central banks in more extreme circumstances, the Fed's discretion over the money supply has often come under pressure following episodes of high inflation. The earliest call for a

[5] For example, see Raskin (2012) and Urban (2017).

privately created digital currency to constrain the money supply is likely by Milton Friedman. In 1999, Friedman foresaw and welcomed the opportunities for an internet-based digital currency to be supplied inelastically without discretion, as described by Raskin and Yermack (2018).

Raskin and Yermack (2018) also argue that either public or private digital currency will ultimately displace the banking system. However, fractional reserve banking based on paying a return on deposits and making loans denominated in bitcoin is emerging. Mastercard has recently won patents, and is applying for additional ones, for methods and systems for a fractional reserve digital currency bank.[6] Over the past several years, a number of platforms have already been providing bitcoin savings accounts that pay interest generated by returns from lending bitcoin for leveraged trading.[7]

Traditional banks have also issued their own digital currency for payments, deposits and withdrawals, including by JP Morgan, a consortium of Japanese banks, and a consortium of Uk and European banks. A wider bitcoin-tied financial system is also developing with corporate bonds denominated in bitcoin issued by Japan's largest financial services provider, Fisco, and bitcoin derivatives including futures, forward contracts and swaps developed by the CME, Goldman Sachs, Morgan Stanley, and other financial institutions. In addition, empirical evidence demonstrates that despite the ability for the growing fintech economy to operate outside of financial intermediation, banking in effect reemerges.[8]

In order to focus on the basic premise of private digital currency created with a fixed supply to prevent the type of inflation that central banks have the discretion to permit, we use a simple model of fiat inflation and private digital currency. The central bank uses a basic form of monetary policy to maximize welfare but may have a bias for shorter term than longer term economic output, which leads to excessive inflation, and which captures the basic time-inconsistency problem of monetary policy. Private digital currency can be created with a fixed supply and is a technology that

[6]See Madore (2018), Suberg (2018), and U.S. Patent and Trademark Office (2018).

[7]See See Kharif (2019), Lielacher (2017), Tomasicchio (2016), Cruz (2015), Johnson (2015), and for a list of such platforms, https://cryptalker.com/bitcoin-interest and https://cryptalker.com/bitcoin-savings.

[8]Balyuk and Davydenko (2018) show that fintech platforms designed for direct peer-to-peer lending are evolving toward becoming essentially online intermediaries in the form of banks that take investment from passive lenders and make active investment decisions for lending to borrowers.

allows for all agents to store and make payments with it.

Digital currency has been much studied recently in the rapidly growing finance and economics literature on fintech, money, and banking.[9] Initial papers on digital currency, banking, and central bank policy highlight several potential benefits and costs of private and public digital currency with a focus on private digital currency competing against monopolist central bank money,[10] public digital currency competing against bank deposits,[11] and competition among private digital currencies.[12] Less attention has been given to concerns about the trade-off of bank lending and financial stability when public and private digital currencies are both available.

The model builds on the theory of nominal bank contracts with fiat central bank money to provide depositors with optimal consumption and financial stability against liquidity and asset risk, as developed by Allen and Gale (1998), Skeie (2004, 2008), Allen et al. (2014) and Allen and Skeie (2018), but they do not consider distortionary fiat inflation, privately created outside money, or digital currencies that enable large scale transactions and storage of money without a banking system.[13] The model also builds on the role of central bank interest rate policy, balance sheet policy and injections of money, demand deposits paid using money in models of real bank deposits, and bank payment systems.[14]

[9]Raskin and Yermack (2018) highlight that debates over private digital currency as competition to fiat money is demanding a resurgence in classical monetary economic theory based on von Mises (1912), Hayek (1976) and Mundell (1998).

[10]Abadi and Brunnermeier (2018) show that because of free entry and distributed ledger fork competition, private digital currencies do not produce profits for the issuer or miners but provide competition that only partially constrains central bank profits arising through monopoly power as a centralized intermediary of fiat money and payments. Schilling and Uhlig (2018) show that for a central bank with commitment to maintain the real value of fiat money, there is exchange-rate indeterminacy for the price of private digital currency.

[11]Andolfatto (2018) finds that interest-bearing central bank digital currency constrains the profit but does not disintermediate monopolistic banks and may even lead to their expansion by providing competition for banks to increase deposit rates. In contrast, Keister and Sanches (2018) find that central bank digital currency increases exchange efficiency in a search economy but crowds out investment by banks that rely on real deposits.

[12]Fernandez-Villaverde and Sanches (2017a,b) find that competition among private digital currencies may implement efficient allocations in a search economy with productive capital but otherwise require unconventional methods for central bank monetary policy.

[13]Conditions for bank runs with nominal bank contracts are also shown by and Schilling et al. (2021) for a price-targeting central bank, and by Diamond and Rajan (2006) and Champ et al. (1996) due to withdrawals of currency out of the banking system based on consumer purchases of goods that must be made with traditional paper currency. Diamond and Rajan (2006) further show that nominal contracts do not protect from bank runs caused by heterogeneous risk in asset returns.

[14]See Freixas et al. (2011), (Martin et al., 2016, 2018), Allen and Gale (1998), Rochet and Vives

In order to focus on the risks of digital currency runs, we shut down other channels affecting digital currency as money that are studied elsewhere. For example, bitcoin and other private digital currencies have displayed extreme price volatility, which limits their acceptance and use. However, this volatility and the widespread adoption of private digital currency as money are part of an economic coordination problem that may be overcome. Several studies argue that the increasing acceptance and use of private digital currency will lead to a more stable value, further supporting its adoption and use.[15] Several papers also tie the extreme volatility of bitcoin to the proof-of-work protocol,[16] which may be fundamentally overcome through alternative protocols, as demonstrated by Saleh (2018b).

Additionally, whether private digital currency can displace central bank fiat money is in part a technological and political question. While bitcoin has gained notoriety for use in black markets and ransomware, such illicit use may be limited because of its public blockchain transaction history that has allowed authorities to find and prosecute illegal users or hack stolen tokens back.[17] Meanwhile, bitcoin is increasingly being used for legitimate transactions.[18] Several less developed countries have struggled between the extremes of officially supporting the adoption of bitcoin and banning its use.[19] However, the development of broader applications of blockchain technology may become so widespread and ubiquitous in the financial system and economy that platforms embedded with private digital currency may require its use.[20]

(2004), Chang and Velasco (2000), Henckel et al. (1999) and Flannery (1996), respectively.

[15]Bolt and van Oordt (2016) show how the price volatility of private digital currency is driven by speculators but decreases as it becomes more widely adopted by consumers and accepted by merchants. Cong et al. (2018) explain the volatility of private digital currency based on the feedback-loop dynamics of it being adopted for transactions. Li and Mann (2018) point to initial coin offerings (ICOs) for investment in private digital currency platforms that can solve the adoption coordination problem. Sockin and Xiong (2018) show that the price and volume of private digital currency transactions act as coordination devices that determine whether there is high, low, or no transactions with the digital currency. Kim (2015) uses empirical evidence based on pre-blockchain based virtual currencies to argue that bitcoin volatility driven by speculators will significantly decrease over time.

[16]Biais et al. (2018b) provide an OLG model and empirical evidence that costly mining determines bitcoin's fundamental value based on the net present value of transactional benefits but also drives large volatility. Pagnotta and Buraschi (2018) show that mining costs being paid in bitcoin amplifies the impact of supply and demand shocks on its price volatility.

[17]See Jawaheri et al. (2018), Meiklejohn (2016), and Bohannon (2016).

[18]See Tasca et al. (2018).

[19]See Hosain (2018).

[20]Applications of blockchain include more efficient smart financial contracting (Cong and He, 2018), managing trading transparency in financial markets to increase investor welfare (Malinova

The paper proceeds with the model of the real economy and the optimal allocation in section 2. Implementation is analyzed with banks for CBDC in section 3 and for private digital currency in section 4. Consumer direct lending to firms is compared in section 5, hard currency is compared in section 6, and section 7 concludes.

2 Real economy

The real economy is similar to that of Diamond and Dybvig (1983) with idiosyncratic and aggregate liquidity demand risk, and additionally with aggregate risk of illiquid investment returns following Allen et al. (2014), but also with the consideration of the early liquidation of illiquid investments required to study the potential for bank runs.

2.1 Real model

Consumers have an endowment and firms have an investment technology. The economy is set up with consumers not having the investment technology, and firms not having an endowment. For investment to occur, consumers will need to lend either directly to firms or through banks as intermediaries, which are introduced in the decentralized economy in the next section.

There are three periods $t = 0, 1, 2$. A continuum $[0, 1]$ of consumers with a unit mass are ex-ante identical, endowed with one unit of goods at $t = 0$, and live for one or two periods. At $t = 1$, a random fraction $\tilde{\lambda}$ of the consumers privately realize they are 'early' (impatient) types with a need to consume at $t = 1$, while the remaining fraction $1 - \tilde{\lambda}$ are 'late' (patient) types who consume at $t = 2$. Consumers have a utility function $u(\cdot)$ that is twice continuously differentiable, strictly concave, satisfies Inada conditions $u'(0) = \infty$ and $u'(\infty) = 0$, and has relative risk aversion $-c \cdot u''(c)/u'(c) \geq 1$ for all consumption levels $c \geq 0$. The coefficient of relative risk aversion (CRRA) is assumed to be a constant for simplicity and for showing solutions in closed form for better insight, however results would be qualitatively the same without this assumption. For notational convenience, the inverse of the CRRA is

and Park, 2017), and the market for, and regulation of, financial reporting and auditing (Cao et al., 2018).

used and is denoted by $\gamma = -u'(c)/u''(c)c$, where $\gamma \in (0, 1]$, $\gamma = 1$ corresponds to $u(c) = \ln c$, and $\gamma \to 0$ corresponds to the limit as the CRRA $\to \infty$.

Firms store g_0 goods at $t = 0$ and g_1 goods at $t = 1$ for a one-period safe and liquid per-unit return of one. Firms invest a_0 of goods as risky illiquid assets at $t = 0$ and liquidate $a_1 \leq a_0$ of the invested goods for a constant per-unit return of $r_1 \in (0, 1)$ at $t = 1$. The remaining $(a_0 - a_1)$ of invested goods not liquidated have a per-unit random return $\tilde{r}_2$ at $t = 2$. Consumers can store goods for a one-period constant per-unit return of $r^\ell \leq 1$, which reflects that if $r^\ell < 1$, consumers cannot store goods as efficiently as firms can.

The random variables $\tilde{\lambda}$ and $\tilde{r}_2$ have a joint distribution with support in the interval $[0, 1] \times (r_1, r_2^{\max})$ and have unconditional means $\bar{\lambda} = E[\tilde{\lambda}] \in (0, 1)$ and $\bar{r}_2 = E[\tilde{r}_2] > 1$, respectively. The aggregate macro state, $\omega = (\lambda, r_2)$, is realized and publicly observable but not verifiable at $t = 1$.

2.2 Optimal allocation

The optimal allocation in the real economy is based on maximizing the expected utility of consumers, since they have the limited resources of endowment goods in the economy, whereas there is an unlimited number of firms available with the investment technology. Consumer expected utility is

$$EU(\beta) = E\left[\lambda u(c_1(\omega)) + (1 - \lambda)\,\beta u(c_2(\omega))\right],$$

where early types consume $c_1(\omega)$, late types consume $c_2(\omega)$, and late types have a discount factor $\beta \leq 1$ relative to early types on their utility function $u(\cdot)$.

At $t = 0$, 'initial investment' by firms includes the a_0 goods invested as assets and g_0 goods stored. At $t = 1$, 'continuing investment' by firms is the $a_0 - a_1(\omega)$ assets not liquidated and $g_1(\omega)$ goods stored. The quantity of goods available for initial investment is q_0 at $t = 0$, and the quantity of goods available for consumption is $q_1(\cdot)$ at $t = 1$ and $q_2(\cdot)$ at $t = 2$:

$$\left.\begin{array}{ll} t{=}0: & q_0 \equiv a_0 + g_0 \\ t{=}1: & q_1(\omega, g_0, a_0, g_1(\cdot), a_1(\cdot)) \equiv g_0 + a_1(\cdot)r_1 - g_1(\cdot) \quad \forall\omega \\ t{=}2: & q_2(\omega, g_0, a_0, g_1(\cdot), a_1(\cdot)) \equiv [a_0 - a_1(\cdot)]r_2 + g_1(\cdot) \quad \forall\omega. \end{array}\right\}$$

An allocation consists of the initial investment a_0 and g_0, the continuing investment $a_0 - a_1(\omega)$ and $g_1(\omega)$, and consumption $c_1(\omega)$ by early types and $c_2(\omega)$ by late types.

The optimization problem for the full-information (first-best) optimal allocation is:

$$\left.\begin{array}{lll} \max\limits_{Q} EU(\beta) = E\left[\lambda u(c_1(\cdot)) + (1-\lambda)\,\beta u(c_2(\cdot))\right] & & \\ \text{s.t.:} \qquad t{=}0: & q_0 \leq 1 & \\ \qquad\qquad t{=}1: & \lambda c_1(\cdot) \leq q_1(\cdot) \quad \forall\omega & \\ \qquad\qquad t{=}2: & (1-\lambda)\,c_2(\cdot) \leq q_2(\cdot) \quad \forall\omega, & \end{array}\right\} \tag{1}$$

which includes the feasibility constraints at periods $t = 0, 1, 2$, and where $Q \equiv \{g_0, a_0, \{g_1(\cdot), a_1(\cdot)\}_\omega\}$. Note that without loss of generality, allocating goods available at $t = 1$ to late consumers is not considered for the optimal allocation.

Consumption is

$$c_1 = \tfrac{q_1}{\lambda} = \tfrac{g_0 + a_1 r_1 - g_1}{\lambda} \quad \forall\omega \tag{2}$$

$$c_2 = \tfrac{q_2}{1-\lambda} = \tfrac{(a_0 - a_1) r_2 + g_1}{1-\lambda} \quad \forall\omega. \tag{3}$$

For brevity of notation, the argument (ω) is typically suppressed in the writing of state-dependent functions after they are first introduced, except when included to provide particular emphasis, and the value of a state-dependent function is typically denoted the same as is the function.

2.3 Analysis

The first order conditions and complementary slackness conditions for the optimization (1) with respect to a_0, $g_1(\cdot)$ and $a_1(\cdot)$ can be expressed as:

$$a_0: \qquad E[u'(c_1^*)] = E[\beta u'(c_2^*) r_2] \tag{4}$$

$$g_1: \qquad \tfrac{u'(c_1^*)}{\beta u'(c_2^*)} = 1 \text{ if } g_1^* > 0 \quad \forall\omega \tag{5}$$

$$a_1: \qquad \tfrac{u'(c_1^*)}{\beta u'(c_2^*)} = \tfrac{r_2}{r_1} \text{ if } a_1^* > 0 \quad \forall\omega, \tag{6}$$

respectively.

These conditions show that the marginal rate of substitution for consumption between early consumers and late consumers equals the relevant marginal rate of transformation for assets and storage in expectation at $t = 0$ and as realized at $t = 1$ as follows.

Equation (4) is an Euler equation reflecting the expected marginal rate of transformation r_2 for assets relative to storage chosen at $t = 0$, which gives optimal risk-sharing in expectation among consumers over the aggregate liquidity and asset risk of the macro state ω, and for consumers' idiosyncratic liquidity risk of being an early type. Note that the term 'rate' is used interchangeably with 'return' throughout the paper to refer to a return (i.e., one plus the interest rate), except when 'interest rate' is specified or clearly intended.

Equations (5) and (6) implicitly define the optimal continuing investment $a_0^* - a_1^*(\omega)$ and $g_1^*(\omega)$ depending on the state ω realized at $t = 1$. The marginal rate of substitution (MRS) between $t = 1$ and $t = 2$ is the marginal utility of consumption for early types relative to that for late types discounted by β. The MRS is denoted by $r^c(\omega)$, which can be expressed as

$$r^c(\omega) \equiv \frac{u'(c_1)}{\beta u'(c_2)} = (\tfrac{c_2}{c_1})^{1/\gamma}/\beta \quad \forall \omega, \tag{7}$$

since $u'(c) = c^{-1/\gamma}$.

The relevant marginal rate of transformation (MRT) for continuing investment between $t = 1$ and $t = 2$ is denoted by $r^k(\omega)$ and depends on whether continuing investment at the margin takes the form of storage, with $g_1 > 0$, or takes the form of asset liquidation, with $a_1 > 0$. Specifically, the MRT for storage at the margin equals the return on storage of 1, $r^k = 1$. The MRT for continuing rather than liquidating assets at the margin is the return on an asset not being liquidated, r_2, relative to being liquidated, r_1: $r^k = \frac{r_2}{r_1}$.

Equations (5) and (6) show that the optimal MRS equals the MRT, $r^{c^*}(\omega) = r^k(\omega)$, when there is either positive storage $g_1^* > 0$ or positive liquidation $a_1^* > 0$, respectively.

For equation (5), $g_1^* > 0$ requires $\frac{u'(c_1^*)}{\beta u'(c_2^*)} = 1$. Consumption for late types is equal to that for early types discounted with the risk-adjusted discount factor β^γ: $c_2^* = \beta^\gamma c_1^*$. Substituting for c_2 with $\beta^\gamma c_1$, equations (2) and (3) show that $g_1^* > 0$ for

$$\lambda < \check{\lambda}(r_2) \equiv \frac{g_0}{g_0 + a_0 r_2/\beta^\gamma} \in (0, 1), \tag{8}$$

and otherwise $g_1^* = 0$ for $\lambda \geq \check{\lambda}(r_2)$. When $\lambda < \check{\lambda}(r_2)$, the aggregate liquidity need λ for early consumers and the asset return r_2 are relatively low. Storage of goods from

$t = 1$ to $t = 2$ enables late consumers to optimally share with early consumers in the overall total amount of goods available over both periods $t = 1$ and $t = 2$, $g_0 + a_0 r_2$.

For equation (6), $a_1^* > 0$ requires $\frac{u'(c_1^*)}{\beta u'(c_2^*)} = \frac{r_2}{r_1}$. which can be written as $c_2^* = c_1^*(\beta \frac{r_2}{r_1})^\gamma$ and substituted into equations (2) and (3) to show that $a_1^* > 0$ for

$$\lambda > \hat{\lambda}(r_2) \equiv \frac{g_0}{g_0 + a_0 r_2/(\beta r_2/r_1)^\gamma} \in (\check{\lambda}(r_2), 1], \tag{9}$$

and otherwise $a_1^* = 0$ for $\lambda \leq \check{\lambda}(r_2)$. When $\lambda > \hat{\lambda}(r_2)$, the liquidity need λ for early consumers and the asset return r_2 are relatively high. Asset liquidation allows for early consumers to optimally share in part of the relative abundance of goods that are available at $t = 2$. As a consequence, for relatively moderate λ and r_2, where $\lambda \in [\check{\lambda}(r_2), \hat{\lambda}(r_2)]$, there is no storage or liquidation, $g_1^* = a_1^* = 0$.

From equations (5) and (6), $r^{c^*} \in [1, \frac{r_2}{r_1}]$, and we can set $r^{k^*}(\omega) = r^{c^*}(\omega)$, with $r^k(\omega)$ interpreted as the shadow MRT for the states ω where $g_1^* = a_1^* = 0$. Hence, for all states the optimal MRS equals the MRT,

$$r^{c^*}(\omega) = r^{k^*}(\omega) \quad \forall \omega, \tag{10}$$

by which relative optimal consumption between late and early types can be expressed in terms of the MRT as

$$\frac{c_2^*}{c_1^*} = (\beta r^{k^*})^\gamma \quad \forall \omega.$$

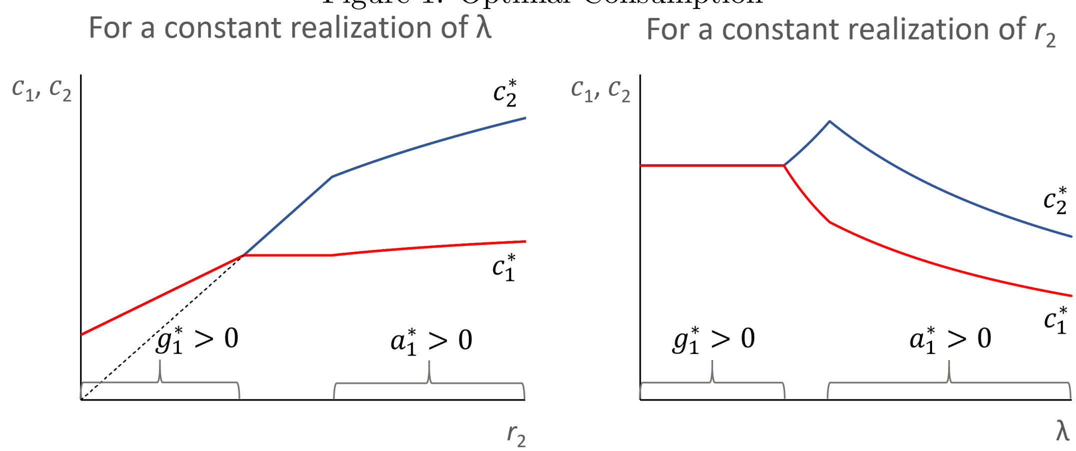
These results are summarized as follows.

Proposition 1 *Optimal consumption and investment equates the optimal marginal rate of substitution between early and late consumers,* $r^{c^*} \equiv \frac{u'(c_1^*)}{\beta u'(c_2^*)}$, *with the optimal marginal rate of transformation,* r^{k^*}. *For relatively low early consumption demand and asset returns, there is optimal storage at* $t = 1$. *For relatively high early consumption demand and asset returns, there is optimal liquidation. Optimal consumption and investment are determined by:*

$$E[u'(c_1^*)] = E[\beta u'(c_2^*) r_2] \tag{11}$$

$$r^{c^*} = r^{k^*} = \begin{cases} 1, & \text{with } g_1^* > 0,\ a_1^* = 0, \quad \text{for } \lambda \in [0, \check{\lambda}), \\ \in [1, \frac{r_2}{r_1}], & \text{with } g_1^* = a_1^* = 0, \quad \text{for } \lambda \in [\check{\lambda}, \hat{\lambda}], \\ \frac{r_2}{r_1}, & \text{with } g_1^* = 0,\ a_1^* > 0, \quad \text{for } \lambda \in (\hat{\lambda}, 1]. \end{cases} \tag{12}$$

Figure 1: Optimal Consumption



The two diagrams in Figure 1 illustrate a comparison of the optimal consumption for early and late consumers, and the ex-interim optimal continuing investment represented by a_1^* and g_1^*. The left diagram shows the realization of r_2 on the x-axis with a fixed realization of λ. The right diagram shows the realization of λ on the x-axis with a fixed realization of r_2.

For a very low realization on the x-axis of r_2 (left diagram) and λ (right diagram), there is positive storage $g_1^* > 0$ at $t = 1$ to provide, in this illustration with $\beta = 1$, equal consumption $c_2^* = c_1^*$. As r_2 increases (left diagram) and as λ increases (right diagram): the storage amount decreases to zero, $g_1^* = 0$, and there is also no liquidation, $a_1^* = 0$; and then eventually there is positive liquidation, $a_1^* > 0$ and in increasing amounts to provide a partial transfer of late consumers' increasing consumption to early consumers. The parameters are $\beta = 1$, $r_1 = 0, .r^{\max} = 6$, and $\gamma = 0.25$ (i.e. CRRA=4). Optimal initial investment is $a_0^* = 0.4$ and $g_0^* = 0.6$. The fixed realized variables are $\lambda = 0.45$ in the left diagram and $r_2 = 3$ in the right diagram.

The model of the real economy, results in proposition 1, and illustration in Figure 1 are similar to Allen et al. (2014) in terms of the random state $(\tilde{\lambda}, \tilde{r}_2)$ and optimal storage of goods g_1^* at $t = 1$, but with the addition of allowing for asset liquidation a_1^* at $t = 1$, which relative to that paper i) increases optimal welfare through greater ex-interim efficient liquidity and asset risk sharing, over λ and r_2, respectively, between early and late consumers, and ii) allows for considering the potential for inefficient bank runs.

3 Fiat money and CBDC

This section introduces fiat money, CBDC and intermediary banks to examine the implementation of the first best allocation in the economy.

3.1 Banks

Banks have no endowment or investment technology of their own but can borrow from the central bank and depositors and can lend to firms. At $t = 0$, a representative bank borrows M^b fiat money from the central bank that the bank lends to firms and/or holds as fiat reserves. Firms use the money borrowed from banks to buy the unit of endowment goods from consumers. Consumers can deposit the money received at the bank and/or hold the money as public digital currency in the form of CBDC at the central bank. Banks, firms and consumers are competitive price-takers.

At the end of $t = 0$, the bank has financial assets L_0 loans and M_0^b fiat reserves, and has liabilities D deposits and M^b borrowing, which are reflected by the bank's budget constraint at $t = 0$,

$$L_0^f + M_0^b \leq D + M^b,$$

where $M^b - M_0^b$ is the bank's net borrowing from the central bank. Throughout the paper, uppercase letters denote nominal values, and lowercase letters denote real values. At $t = 1$, the bank pays the return R_1^d on the $\lambda' \in [\lambda, 1]$ fraction of deposits withdrawn. The bank receives return $R_1^f(\omega)$ on loans and $R_1^m(\omega)$ on reserves, lends $L_1^f(\omega)$, and holds $M_1^b(\omega)$ reserves. Banks can further borrow from the central bank at $t = 1$, which would be indicated by $M_1^b(\omega) < 0$. At $t = 2$, the bank pays return $R_2^d(\omega)$ on the remaining $1 - \lambda'$ deposits withdrawn, receives return $R_2^f(\omega)$ on loans and $R_2^m(\omega)$ on reserves, and pays the cumulative return $R_1^m(\omega)R_2^m(\omega)$ to the central bank on the original $t = 0$ borrowing of M^b.

Because of free entry for banks, the bank optimization is to maximize consumer expected utility (equivalent to depositor expected utility), subject to the bank budget constraints at each period:

$$\left.\begin{array}{lll}\max\limits_{Q^b} EU(\beta) = E\left[\lambda u(c_1(\cdot)) + (1-\lambda)\,\beta u(c_2(\cdot))\right] & & \\ \text{s.t.:} & t{=}0: & L_0^f + M_0^b \leq D + M^b \\ & t{=}1: & \lambda' D R_1^d \leq (L_0^f R_1^f - L_1^f) + M_0^b R_1^m - M_1^b \quad \forall\omega \\ & t{=}2: & (1-\lambda') D R_2^d \leq L_1^f R_2^f + M_1^b R_2^m - M^b R_1^m R_2^m \quad \forall\omega,\end{array}\right\} \tag{13}$$

where $Q^b \equiv \{D, L_0^f, M^b, M_0^b, \{L_1^f(\cdot), M_1^b(\cdot)\}_\omega\}$. The return R_1^d paid on deposit withdrawals at $t = 1$ is a fixed nominal amount not contingent on the state ω or on consumer types, which are not verifiable. $R_2^d(\omega)$ is dependent of ω, which reflects that depositors withdrawing at $t = 2$ are the residual claimants on the bank. $R_t^f(\omega)$ and $R_t^m(\omega)$ for $t = 1, 2$ are also dependent on ω.

At each period $t = 0, 1$, the opportunity cost to the bank of lending L_t^f is the return on reserves M_t^b. This requires the return on loans to equal the return on reserves set by the central bank in accord with the bank's first order condition with respect to M_t^b for $t = 0, 1$.

Lemma 1 *The bank lending rate equals the central bank policy rate on reserves:*

$$R_t^f(\omega) = R_t^m(\omega) \quad \textit{for } t = 1, 2 \textit{ and } \forall\omega. \tag{14}$$

The bank's three budget constraints can be consolidated into a single budget constraint, which after substituting for R_t^f with R_t^m for $t = 1, 2$ and simplifying, can be written as

$$(1-\lambda') R_2^d \leq (R_1^f - \lambda' R_1^d) R_2^f \quad \forall\omega. \tag{15}$$

3.2 Consumers

At $t = 0$, the representative consumer holds D deposits and M_0^c CBDC out of the proceeds of selling her 1 unit endowment of goods to firms:

$$D + M_0^c \leq 1 \times P_0, \tag{16}$$

where P_0, $P_1(\omega)$ and $P_2(\omega)$ are the prices of goods in terms of fiat money as numeraire for $t = 0, 1, 2$, respectively.

At $t = 1$, an early consumer uses the returns on deposits and CBDC to buy and consume $c_1(\omega)$ goods:

$$c_1(\omega) \equiv \tfrac{DR_1^d + M_0^c R_1^c}{P_1(\cdot)} \quad \forall \omega, \tag{17}$$

where R_1^c is the per-unit return on CBDC at $t = 1$.

At $t = 1$, a late consumer holds $M_1^c(\omega) \geq 0$ CBDC at $t = 1$ for a per-unit return of $R_2^c(\omega)$ at $t = 2$. The late consumer withdraws her deposit either at $t = 1$ or at $t = 2$ indicated by $w(\omega) \in \{0, 1\}$, where $w(\cdot) = 1$ is a withdrawal at $t = 1$ and $w(\cdot) = 0$ is a withdrawal at $t = 2$. The withdrawal choice $w(\cdot)$ is a binary $\{0, 1\}$ rather than a continuous fraction within $[0, 1]$ for simplicity and without loss of generality.

The late consumer buys $c_1^\ell(\omega)$ goods (if any) at $t = 1$ using early withdrawal returns (wDR_1^d) and CBDC returns ($M_0^c R_1^c$) less M_1^c CBDC held at $t = 1$. The late consumer buys $c_2^\ell(\omega)$ goods at $t = 2$ using returns on remaining deposits, $(1-w)DR_2^d$, and on CBDC, $M_1^c R_2^c$. The amount of goods *bought* by the late type at $t = 1$ and $t = 2$ are

$$\left.\begin{array}{ll} c_1^\ell(\omega) \equiv \frac{wDR_1^d + M_0^c R_1^c - M_1^c(\cdot)}{P_1(\cdot)} & \forall \omega \\ c_2^\ell(\omega) \equiv \frac{(1-w)DR_2^d(\cdot) + M_1^c(\cdot) R_2^m(\cdot)}{P_2(\cdot)} & \forall \omega, \end{array}\right\} \tag{18}$$

respectively. The amount of goods *consumed* by the late type at $t = 2$ is

$$c_2(\omega) \equiv r^\ell c_1^\ell(\cdot) + c_2^\ell(\cdot) \quad \forall \omega,$$

where r^ℓ is the real per-unit return on the c_1^ℓ goods she stores from $t = 1$ to $t = 2$.

We set $r^\ell \equiv \beta^\gamma \leq 1$, such that $r^\ell = 1$ if $\beta = 1$, and $r^\ell < 1$ if $\beta < 1$, since $\gamma \in (0, 1]$. This parameterization of r^ℓ is used so that in states where there is an optimal storage of goods between $t = 1$ and $t = 2$, such that $c_2^* = \beta^\gamma c_1^*$, late consumers are indifferent between (i) receiving at $t = 1$ an amount $c_1^\ell = c_1^*$ of goods for a self-storage return of $r^\ell c_1^\ell = \beta^\gamma c_1^\ell$ goods to consume at $t = 2$, versus instead (ii) receiving at $t = 2$ an amount $c_2^\ell = \beta^\gamma c_1^*$ of goods to consume at $t = 2$.

The consumer optimization is:

$$\left.\begin{array}{l} \max\limits_{Q^c} EU(\beta) = E\left[\lambda u(c_1(\cdot)) + (1-\lambda)\,\beta u(c_2(\cdot))\right] \\ \text{s.t.:} \qquad t{=}0: \quad \text{Equation (16)}, \end{array}\right\} \tag{19}$$

where $Q^c \equiv \{D, M_0^c, \{M_1^c(\cdot), w(\cdot)\}_\omega\}$. Note that strategic-acting late consumers and hence strategic-based bank runs are not considered, but allowing for these would not change the results.

In order to allow for a continuous amount of aggregate early withdrawals by late consumers, we index late consumers by $i \in [\lambda, 1]$ and each of their early withdrawal choices by $w^i(\omega)$. The aggregate fraction of consumers withdrawing at $t = 1$ is $\lambda'(\omega) \equiv \lambda + \int_{i\in[\lambda,1]} w^i(\omega)di$ and at $t = 2$ is $1 - \lambda'(\omega)$.

3.3 Firms

The representative firm invests a_0 goods as assets and g_0 goods as storage out of the q_0 goods the firm buys with L_0^f loans at $t = 0$, where $q_0 = a_0 + g_0$. At $t = 1$, the firm sells $q_1(\cdot) = g_0 + a_1 r_1 - g_1$ goods to repay its borrowing that is not rolled over, $L_0^f R_1^f - L_1^f$. At $t = 2$, the firm sells $q_2(\cdot) = (a_0 - a_1)r_2 + g_1$ goods to repay its rolled over borrowing, $L_1^f R_2^f$, and consumes any remaining goods $c_2^f(\omega)$ as profit. The firm maximizes its expected profit:

$$\left.\begin{array}{lll} \max\limits_{Q^f} E[c_2^f] & & \\ \text{s.t.:} & t\text{=}0\text{:} & q_0 P_0 \leq L_0^f \\ & t\text{=}1\text{:} & L_0^f R_1^f - L_1^f \leq q_1 P_1 \quad \forall\omega \\ & t\text{=}2\text{:} & L_1^f R_2^f \leq (q_2 - c_2^f) P_2 \quad \forall\omega, \end{array}\right\} \tag{20}$$

which includes the firm's budget constraint for each period $t = 0, 1, 2$, and where $Q^f \equiv \{g_0, a_0, L_0^f, \{g_1(\cdot), a_1(\cdot), L_1^f(\cdot)\}_\omega\}$. While in principle firms could hold digital currency, in the form of wholesale CBDC with a return equal to that paid on bank fiat reserves, in equilibrium they would not and so is not considered.

3.4 Central bank

The central bank sets the state-contingent monetary policy rates on fiat reserves and CBDC to affect initial and ongoing investment in order to optimize consumer expected utility, $EU(\beta^{cb})$, as a function of the central bank's discount factor $\beta^{cb} \leq \beta^\ell$.

If $\beta^{cb} < \beta^\ell$, the central bank has a bias for relatively higher short-term than long-term output and consumption. This is motivated by and captures in reduced form the standard time-inconsistency problem for central banks without a commitment device to use monetary policy to boost the economy in the short term ($t = 1$) at the expense of higher inflation and a worse-off economy in the long term ($t = 2$), as will be shown below in section 3.8.

The central bank's optimization is:

$$\max_{\{R_t^m(\cdot), R_t^c(\cdot)\}_{t\in\{1,2\},\omega}} EU(\beta^{cb}). \tag{21}$$

3.5 Equilibrium

An equilibrium is defined as follows.

Definition 1 *An equilibrium consists of prices and returns*

$$\{P_0, P_t(\cdot), R_t^j(\cdot)\}_{j\in\{d,f,m,c\},t\in\{1,2\},\omega}\ ,$$

and quantities Q^b, Q^c, Q^f, such that the following conditions are satisfied.
i) The quantities Q^b, Q^c, Q^f satisfy the optimizations for banks (13), consumers (19) and firms (20), respectively,
ii) The returns $\{R_t^m(\cdot), R_t^c(\cdot)\}_{t\in\{1,2\},\omega}$ satisfy the central bank optimization (21), and
iii) Markets clear for deposits D at $t = 0$, loans to firms L_0^f at $t = 0$ and $L_1^f(\omega)$ $\forall\omega$ at $t = 1$, and goods at $t \in \{0, 1, 2\}$:

$t = 0$: $q_0 = 1$,
$t = 1$: $\lambda c_1(\omega) + (1-\lambda)c_1^\ell(\omega) = q_1(\omega) \quad \forall\omega$,
$t = 2$: $(1-\lambda)c_2(\omega) = q_2(\omega) \quad \forall\omega$.

3.6 Efficiency

We first analyze the equilibrium without CBDC, with $M_0^c \equiv M_1^c(\cdot) \equiv 0\ \forall\omega$, to show how the the central bank can set optimal policy rates on reserves to implement the optimal allocation without traditional bank runs.

Investment For $M_0^c = M_1^c = 0$, prices are given by

$$P_1 = \tfrac{\lambda' D R_1^d}{q_1} \quad \forall\omega \tag{22}$$
$$P_2 = \tfrac{(1-\lambda')D R_2^d}{q_2} \quad \forall\omega, \tag{23}$$

which for each of the two periods reflects the amount of money withdrawn and spent by consumers in the numerator divided by the amount of goods sold by firms in the denominator.

The real return on loans, $r_t^f(\omega)$, is equal to the nominal return on loans divided by inflation, as based on the Fisher equation:

$$r_t^f(\omega) \equiv \frac{R_t^f(\omega)}{\Pi_t(\omega)} \text{ for } t \in \{1, 2\} \text{ and } \forall\omega, \tag{24}$$

where inflation is

$$\Pi_t(\omega) \equiv \frac{P_t}{P_{t-1}} \text{ for } t = 1, 2 \text{ and } \forall\omega, \tag{25}$$

such that $\Pi_1(\omega)$ is the one-period inflation at $t = 1$ and $\Pi_2(\omega)$ is the one-period inflation at $t = 2$. The firm's three budget constraints can be combined into a single consolidated budget constraint, solved for firm profit c_2^f, and expressed in real terms as:

$$c_2^f(\cdot) \leq q_2(\cdot) + [q_1(\cdot) - q_0(\cdot)r_1^f]r_2^f \quad \forall\omega. \tag{26}$$

The firm's first order conditions and complementary slackness conditions can be expressed as:

$$\begin{aligned}
g_0 :& \quad E[(r_1^f - 1)r_2^f]g_0 = 0 \\
a_0 :& \quad E[r_1^f r_2^f]a_0 = \bar{r} \\
g_1 :& \quad (r_2^f - 1)g_1 = 0 \quad \forall\omega \\
a_1 :& \quad (r_2^f - \tfrac{r_2}{r_1})a_1 = 0 \quad \forall\omega.
\end{aligned}$$

The first two equations together require for an interior solution:

$$E[r_2^f] = \bar{r}. \tag{27}$$

The last two equations together require that r_2^f, the real return on loans at $t = 2$ and henceforth referred to as the real rate unless otherwise specified, equals r^k, the MRT between $t = 1$ and $t = 2$:

$$r_2^f(\omega) = r^k(\omega) \quad \forall\omega \tag{28}$$

At $t = 1$, if there is positive storage $g_1 > 0$, then $r_2^f = r^k = 1$. If there is liquidation, $a_1 > 0$, then $r_2^f = r^k = \frac{r_2}{r_1}$. Thus if neither, $g_1 = a_1 = 0$, then $r_2^f = r^k \in [1, \frac{r_2}{r_1}]$. These conditions are summarized as:

$$r_2^f = r^k = \begin{cases} 1 & \Rightarrow g_1 \geq 0,\ a_1 = 0 \\ \in (1, \frac{r_2}{r_1}) & \Rightarrow g_1 = a_1 = 0 \quad \forall \omega \\ \frac{r_2}{r_1} & \Rightarrow g_1 = 0,\ a_1 \geq 0 \quad \forall \omega. \end{cases} \tag{29}$$

To solve for r_t^f for each of $t \in \{1, 2\}$, substituting for P_t from equations (22) and (23) into the equation for inflation (25), which is then substituted into the real rate equation (24), and simplifying gives

$$\begin{aligned} r_1^f &= \frac{R_1^f}{R_1^d}\frac{q_1}{\lambda'}P_0 \\ r_2^f &= \frac{\lambda' R_1^d R_2^f}{(1-\lambda')R_2^d}\frac{q_2}{q_1}. \end{aligned} \tag{30}$$

Subsituting for $(1-\lambda')R_2^d$ with $(R_1^f - \lambda' R_1^d)R_2^f$ from the bank's consolidated budget constraint (15) into the denominator of P_2, and then substituting for the resulting P_2 along with P_1 for Π_2 into the real rate equation $r_2^f = \frac{R_2^f}{\Pi_2}$, and simplifying, the real rate can be expressed as

$$r_2^f = (\frac{\lambda'}{R_1^f/R_1^d - \lambda'})\frac{q_2}{q_1}. \tag{31}$$

Firms have zero consumption $c_2^f = 0$ for all states ω, which can be seen by substituting for r_1^f and r_2^f from equations (30) and (31) into equation (26) and simplifying.

At $t = 1$, continuing investment has positive storage if $g_1 > 0$ for $r_2^f = 1$, which from equation (31) holds with

$$g_1 = g_0 - \lambda'(R_1^f/R_1^d)(g_0 + a_0 r_2) > 0 \tag{32}$$

when λ' and r_2 are relatively low, written as

$$\lambda' < \check{\lambda}^f(r_2) \equiv \frac{g_0(R_1^f/R_1^d)}{g_0 + a_0 r_2}. \tag{33}$$

There is liquidation if $a_1 > 0$ for $r_2 = \frac{r_2}{r_1}$, which from equation (31) holds with

$$a_1 = [\lambda'(R_1^d/R_1^f)(g_0 + a_0 r_1) - g_0]/r_1 > 0 \tag{34}$$

when λ' is relatively high, written as

$$\lambda' > \hat{\lambda}^f(r_2) \equiv \frac{g_0(R_1^f/R_1^d)}{g_0 + a_0 r_1}. \tag{35}$$

There is no storage or liquidation, $g_1^* = a_1^* = 0$, for relatively moderate λ and r_2, written as $\lambda \in [\check{\lambda}^f, \hat{\lambda}^f]$, where $\hat{\lambda}^f$ is used an abbreviation for $\hat{\lambda}^f(r_2)$.

After substituting for the prices in equations (22) and (23) into the consumption equations (17) and (18) then simplifying, consumption for those withdrawing at $t = 1$ and $t = 2$ is

$$c_1 = \tfrac{DR_1^d}{P_1} = \tfrac{q_1}{\lambda'} \tag{36}$$
$$c_2 = \tfrac{DR_2^d}{P_2} = \tfrac{q_2}{1-\lambda'}, \tag{37}$$

respectively. Hence, consumption is determined by λ', q_1 and q_2, where the latter two are determined through a_t and g_t for $t = 0, 1$ by r^k.

The real rates r_1^f and r_2^f can be expressed in terms of depositors' consumption by substituting for $q_1 = \lambda' c_1$ and $q_2' = (1 - \lambda')c_2$ solved from the consumption equations (36) and (37) into the real rate equations (30) and (31), respectively, and simplifying:

$$r_1^f = \frac{R_1^f}{R_1^d} c_1 P_0 \tag{38}$$
$$r_2^f = \frac{(1-\lambda')c_2}{(R_1^f/R_1^d - \lambda')c_1}. \tag{39}$$

Relative consumption between late and early types can be expressed in terms of the real rate by rearranging equation (39) for

$$\frac{c_2}{c_1} = \frac{(R_1^f/R_1^d - \lambda')}{(1-\lambda')} r_2^f, \tag{40}$$

which since the real rate equals the MRT, $r_2^f = r^k$, can also be expressed as

$$\frac{c_2}{c_1} = \frac{(R_1^f/R_1^d - \lambda')}{(1-\lambda')} r_2^k. \tag{41}$$

The MRS $r^c \equiv \frac{u'(c_1)}{\beta u'(c_2)}$ that obtains can be expressed in terms of the MRT by substituting for $\frac{c_2}{c_1}$ from equation (39):

$$r^c(R_1^d, R_1^f, r^k) = \left(\tfrac{(R_1^f/R_1^d - \lambda')}{(1-\lambda')}\right)^{1/\gamma} r_k^{1/\gamma}/\beta. \tag{42}$$

Rates on reserves Since the central bank has monopoly power over fiat money to act as a price-setter on interest rates for reserves and CBDC, and nominal rates for loans equal those for bank reserves, $R_t^f = R_t^m$ for $t = 1, 2$, the central bank chooses its policy rates to equate the MRS $r^c(R_1^d, R_1^f, r^k)$ given by equation (42) with the optimal MRS $r^{c^*}(\beta)$ given by equation (10). This can be expressed as the optimal return that the central bank pays on reserves relative to the deposit rate at $t = 1$:

$$\frac{R_1^{m^*}}{R_1^{d^*}} = \lambda' + (1 - \lambda')(\beta r^k)^\gamma / r^k \leq 1. \tag{43}$$

The bank's consolidated budget constraint equation (15) binds, and after substituting for R_t^f with R_t^m for $t = 1, 2$, can be expressed as the optimal return that the central bank pays on reserves relative to the deposit rate at $t = 2$:

$$\frac{R_2^{m^*}}{R_2^{d^*}} = \frac{(1 - \lambda')R_1^{d^*}}{(R_1^{m^*}/R_1^{d^*} - \lambda')} = r^k R_1^{d^*}/(\beta r^k)^\gamma \geq R_1^{d^*}. \tag{44}$$

The following proposition states that before considering the potential for consumers to hold CBDC, the central bank with bank lending and firm investment implements the optimal allocation as a function of the central bank's discount factor $\beta = \beta^{cb}$.

Proposition 2 *In the absence of CBDC, the central bank sets the policy rates* $R_t^{m^*}(\omega)$ *and* $R_t^{c^*}(\omega)$ *such that with optimal inflation* $\Pi_t^*(\omega)$*, bank lending and firm investment implement the optimal allocation (as a function of* $\beta = \beta^{cb}$*) of initial and continuing investment and consumption* $\{a_t^*(\omega), g_t^*, c_t^*(\omega)\}_{t=1,2}$.

Traditional bank runs Proposition (2) reflects that there are no traditional inefficient bank runs from late consumers withdrawing and buying goods at $t = 1$. With $M_0^c = M_1^c = 0$, the first order condition with respect to $w(\omega)$ of the consumer optimization (19) for a late consumer to withdraw early to buy goods at $t = 1$, $w(\cdot) = 1$, requires $r^\ell c_1^\ell \geq c_2^\ell$, i.e. $\frac{DR_1^d}{P_1(\cdot)} \geq \frac{DR_2^d(\cdot)/r^\ell}{P_2(\cdot)}$, which is hence the requirement for $\lambda' > \lambda$. The condition can be expressed as the nominal return on holding deposits being less than inflation between $t = 1$ and $t = 2$, written as $\frac{R_2^d(\cdot)/r^\ell}{R_1^d} < \Pi_2(\cdot)$.

Corollary 1 *There are no traditional bank runs by late consumers withdrawing and buying goods early at* $t = 1$ $\forall\omega$.

Consider the scenario of potential withdrawals by some or all late consumers $i \in [\lambda, 1]$ with $w = 1$ to buy goods at $t = 1$ for any particular realized state $\omega = (\lambda, r_2)$. Total early withdrawals in this state at $t = 1$ is $\lambda' \in [\lambda, 1]$. This is equivalent to the scenario of an alternative realized state with a fraction of early consumers λ' and the same asset return r_2, $\omega' = (\lambda', r_2)$, in which there are no early withdrawals by late consumers: $w = 0$ for all i. In each scenario, consumption for early types and late types (if any) withdrawing at $t = 1$ is $c_1^*(\lambda', r_2)$, and consumption for late types withdrawing at $t = 2$ is $c_2^*(\lambda', r_2)$. Consumption for late types withdrawing at $t = 2$ rather than at $t = 1$ is strictly greater when $\lambda' > \check{\lambda}(r_2)$, $c_2^*(\lambda', r_2) > c_1^*(\lambda', r_2)$, and is equal when $\lambda' \leq \check{\lambda}(r_2)$, $c_2^*(\lambda', r_2) = c_1^*(\lambda', r_2)$.

Consumption per unit withdrawn at $t = 1$ for late consumers equals that for early consumers: $c_1^\ell = c_1^* \leq c_2^*$, with total consumption for late consumers of $c_2(w) = wc_1^* + (1 - w)c_2^*$. For $\lambda > \hat{\lambda}(r_2)$, $c_2 > c_1$, so $w = 0$. For $\lambda < \hat{\lambda}(r_2)$, $c_2 = c_1$, so $w \in [0, 1]$ is not determined, has no impact on real allocations, and without loss of generality can be set to $w = 0$. Hence, late consumers never withdraw early: $w = 0$.

The bank can pay all withdrawals at $t = 1$ for all states ω. Since the optimal consumption obtains, $c_2 = \frac{R_2^d}{P_2} = c_2^*$ and $c_1 = \frac{R_1^d}{P_1} = c_1^*$, with $c_2^* \geq c_1^*$, late consumers do not withdraw early to buy goods at $t = 1$. Since there are no bank defaults, late consumers do not withdraw early.

3.7 Bank disintermediation and CBDC runs

Now we consider CBDC that consumers can hold at $t = 0$ and $t = 1$, $M_0^c \geq 0$ and $M_1^c \geq 0$, respectively. We analyze the potential for bank disintermediation, defined as consumers initially holding CBDC rather than bank deposits at $t = 0$, and the potential for CBDC runs, defined as late consumers running on the bank by withdrawing deposits to hold CBDC at $t = 1$. The next lemma follows from the first order conditions with respect to M_0^c and $M_1^c(\cdot)$ of the consumer optimization (19).

Lemma 2 *Consumers hold CBDC rather than bank deposits at* $t = 0$ *if* $R_1^c > R_1^d$, *and at* $t = 1$ *if* $R_2^c(\omega) > \frac{R_2^d(\omega)}{R_1^d}$.

Following this lemma, the central bank must set $R_1^c \leq R_1^d$ and $R_2^c(\cdot) \leq \frac{R_2^d(\cdot)}{R_1^d}$ to prevent bank disintermediation and runs. However, equations 43 and 44 require that $R_1^m \leq R_1^d$ and $R_2^m \geq R_1^d R_2^d$.

Proposition 3 *The central bank must set appropriate state-contingent interest rates paid on reserves, relative to interest rates paid on CBDC that can be either fixed (including zero rates) or state-contingent, to prevent the disintermediation of banks at $t = 0$, which requires $R_1^m \leq R_1^c$, and to prevent CBDC runs at $t = 1$, which requires $R_2^m \geq R_1^c R_2^c$.*

3.8 Short-term central bank

If the central bank has a short-term bias, $\beta^{cb} < \beta^{\ell}$, the central bank sets a lower short-term nominal policy rate than optimal, $R_1^m(\omega, \beta^{cb}) < R_1^{m^*}(\omega, \beta^{\ell})$. To maintain equilibrium, the central bank must counteract the lower $t = 1$ rate with a higher $t = 2$ rate, $R_2^m(\omega, \beta^{cb}) > R_2^{m^*}(\omega, \beta^{\ell})$. These nominal rates drive lower short-term inflation $\Pi_1 < \Pi_1^*$ and real rate $r_1^f < r_1^{f^*}$ at $t = 1$, and higher long-term inflation $\Pi_2 > \Pi_2^*$ and real rate $r_2^f > r_2^{f^*}$ at $t = 2$, which satisfy the Fisher equation $\Pi_t r_t^f = R_t^m(\omega, \beta^{cb})$ at each period $t \in \{1, 2\}$. Firms lower continuing investment at $t = 1$ in assets, $a_0 - a_1 < a_0 - a_1^*$, and in storage, $g_1 < g_1^*$, and may also lower initial investment in assets at at $t = 0$, $a_0 < a_0^*$. The result is higher overall fiat inflation, $\Pi_1\Pi_2 > \Pi_1^*\Pi_2^*$. There is relatively higher output and consumption at $t = 1$ at the expense of lower output and consumption $t = 2$: $q_1 > q_1^*$, $c_1 > c_1^*$, $q_1 > q_1^*$ and $c_1 > c_1^*$.

The central bank short-term bias can take two different forms that are analyzed in turn. One form is that the central bank's bias $\beta^{cb} < \beta^{\ell}$ comes as a surprise to the public at period $t = 1$, after initial investment decisions are made at period $t = 0$. The second form is that the central bank's bias is known by the public at period $t = 0$.

Surprise excessive inflation For the first form of bias, the public expects at $t = 0$ that the central bank has a discount factor $\beta^{cb} = \beta^{\ell}$ and will set the optimal policy resulting in optimal inflation Π_2^* and real rates r_1^* and r_2^*. Firms choose a_0^* as their initial asset investment. When the central bank surprises the public by setting a lower than expected policy rate $R_1^m(\cdot)$, firms' ex-interim state-dependent cutoffs for storage and liquidation at $t = 1$ are lower than optimal, $\check{\lambda}^{cb}(r_2, a_0^*) < \check{\lambda}^*(r_2, a_0^*)$ and $\hat{\lambda}^{cb}(r_2, a_0^*) < \hat{\lambda}^*(r_2, a_0^*)$, respectivevly. Ex-interim storage is lower and liquidation is higher than optimal, $g_1^{cb}(\omega, a_0^*) < g_1^*(\omega, a_0^*)$ and $a_1^{cb}(\omega, a_0^*) > a_1^*(\omega, a_0^*)$, in the states where ex-interim storage and liquidation occur, respectively. The ex-interim storage and liquidation cutoffs and amounts are inefficient relative to the optimum conditional on either the central bank or late consumer discount factor.

Consumption is weakly greater for early consumers and lower for late consumers than optimal, $c_1 \geq c_1^*$ and $c_2 \leq c_2^*$, respectively, and strictly so in the states $\lambda < \check{\lambda}^{cb}(r_2, a_0^*)$ and $\lambda > \hat{\lambda}^{cb}(r_2, a_0^*)$ with storage and liquidation, respectively.

Expected excessive inflation For the second form of central bank bias, the public knows the central bank's discount factor $\beta^{cb} < \beta^{\ell}$ and anticipates the lower short-term rates and higher long-term inflation Π_2^{cb}. Firms have lower initial asset investment and store excessive goods at $t = 0$, $a_0^{cb} < a_0^*$ and $g_0^{cb} > g_0^*$. The ex-interim state-dependent cutoffs for storage and liquidation are still lower than optimal, though by not as much as in the unexpected case, and ex-interim storage is still lower and liquidation still higher than optimal in the states when these occur. However, the policy rates, and initial and ongoing investment, are efficient conditional on the central bank discount factor β^{cb}.

Consumption is strictly greater for early consumers and lower for late consumers than optimal conditional on the late consumer discount factor β^{ℓ}, but consumption is optimal conditional on the central bank discount factor β^{cb}: $c_1^{cb} = c_1^*(\beta^{cb}) > c_1^*(\beta^{\ell})$ and $c_2^{cb} = c_2^*(\beta^{cb}) < c_2^*(\beta^{\ell})$.

Proposition 4 *For either an unexpected or expected central bank short-term bias of $\beta^{cb} < \beta^{\ell}$, there is excessive inflation at $t = 2$ of $\Pi_2 > \Pi_2^*$ through the central bank setting a lower short-term nominal rate $R_1^m < R_1^{m^*}$ at $t = 1$. Output q_t and consumption c_t are lower than optimal at $t = 1$ and higher than optimal at $t = 2$.*

We proceed by assuming that the central bank's discount factor $\beta^{cb} \leq \beta^{\ell}$ is known by the public at $t = 0$, such that excessive fiat inflation is fully anticipated when the central bank has a short-term bias.

3.9 Discussion

Efficiency For any β, bank lending and firm investment determine the optimal state-contingent inflation Π_1^* and Π_2^* . The market provides the optimal rationing of goods between early and late consumers through the optimal quantity of goods sold by firms at each period, $q_1^* = \frac{g_0^* + a_1^* r_1 - g_1^*}{\lambda} = \frac{c_1^*}{\lambda}$ and $q_2^* = \frac{(a_0^* - a_1^*) r_2 + g_1^*}{1-\lambda} = \frac{c_2^*}{1-\lambda}$.

For a moderate realization of ω given by $\lambda \in [\check{\lambda}(r_2), \hat{\lambda}(r_2)]$, the equilibrium price levels at $t = 1$ and $t = 2$ are moderate, with $P_1 = \frac{\lambda D R_1^d}{g_0^*}$ and $P_2 = \frac{(1-\lambda) D R_2^d}{a_0^* r_2}$, and the real rate is the optimal $r_2^{f^*} \in [1, \frac{r_2}{r_1}]$. Firms sell at $t = 1$ all of their goods stored from $t = 0$ and sell at $t = 2$ the returns on all their assets.

For a low realization of the state, $\lambda < \check{\lambda}(r_2)$, there is downward pressure on P_1 and upward pressure on P_2, with optimal real rate $r_2^{f^*} = 1$. With fewer early consumers, the amount of money spent for goods is reduced at $t = 1$ and increased at $t = 2$. With lower returns, fewer goods produced by assets are available to sell at $t = 2$. Firms respond to these market prices by storing the optimal amount g_1^* of their goods at $t = 1$ to sell at $t = 2$, which provides for equal consumption among early consumers withdrawing at $t = 1$ and late depositors withdrawing at $t = 2$.

Conversely, for a high realization of the state, $\lambda > \hat{\lambda}(r_2)$, there is relative upward pressure on P_1 and downward pressure on P_2, with optimal real rate $r_2^{f^*} = \frac{r_2}{r_1}$. Firms respond by liquidating the optimal amount a_1^* of their assets to sell additional goods at $t = 1$, which results in the marginal rate of substitution between early and late consumers equal to the marginal rate of transformation between asset returns and liquidation returns: $\frac{u'(c_1^*)}{u'(c_2^*)} = \frac{r_2}{r_1}$.

Financial stability Since deposits pay out nominal amounts, the bank can pay fixed promises in terms of money as numeraire, while the real return on deposits provides consumption contingent on the aggregate state for early and late types.

Inflation on fiat money that enables the sharing of macro risks between early and late consumers also enables financial stability against the two primary risks inherent in the banking system. One risk is solvency-based bank runs from the potential insolvency of the banking system in the case of low real returns on assets, r_2. The second risk is liquidity-based bank runs from the potential illiquidity of the banking system in the case of a large fraction of early consumers, λ, or early withdrawals by late consumers, $w > 0$.

First, consider the risk of insolvency in the case of low realizations of r_2. When there is a real loss on assets, $r_2 < 1$, the per capita consumption that is available to depositors is less than one. To avoid bank runs and complete asset liquidation when there is an asset loss, $r_2 < 1$, there is sufficient inflation for banks to remain solvent on their nominal deposit contracts. This inflation decreases the real cost of firm and bank liabilities. P_2 increases due to the reduction in goods available to sell at $t = 2$.

With low r_2 and many late consumers $1-\lambda$, firms optimally respond to what would otherwise be even higher inflation caused by the low return $r_2 < 1$ and high $1 - \lambda$ by storing goods at $t = 1$ to sell at $t = 2$. Specifically, $g_1 > 0$ for $\lambda < \check{\lambda}(r_2) = \frac{g_0^*}{g_0^* + a_0^* r_2}$, which allows late consumers to have consumption c_2^* that is optimally equal to that of early consumers, c_1^*, by withdrawing at $t = 2$ and not running on the bank.

Late consumers do not receive any greater consumption by running the bank to buy goods at $t = 1$. Moreover, banks are effectively hedged on their nominal deposit liabilities at $t = 2$. The equilibrium price level at $t = 2$ remains elevated even with the counterbalancing effect of firms selling more goods at $t = 2$. The elevated price level implies that the real cost of bank deposit liabilities at $t = 2$ falls enough that banks do not default.

Second, consider the risk of the bank defaulting when there is a large realization of early consumers, λ, and/or early withdrawals by late consumers, $w \in (0, 1]$. P_1 increases from the larger amounts of money spent for goods at $t = 1$. This leads firms to liquidate a greater amount of assets to sell additional goods at $t = 1$. While additional goods sold provides a partial counterbalancing effect on the price level, P_1 is still sufficiently elevated such that firms do not default on their loans to banks, and banks do not default on paying withdrawals. Banks continue to rollover loans to firms, which enables firms to only liquidate assets to the extent that it is profit-maximizing according to selling goods at $t = 1$ relative to at $t = 2$, and which provides the optimal allocation between consumption from withdrawals at $t = 1$ relative to at $t = 2$. Consumption from withdrawing and buying goods at $t = 2$ relative to at $t = 1$, $\frac{c_2}{c_1}$, actually increases in the fraction of early consumers, λ, and early withdrawals by late consumers, w, because of the relatively higher nominal deposit return, $\frac{R_2^d}{R_1^d}$, and lower price level $\frac{P_2}{P_1}$, which reflects the relatively higher asset return, r_2 when there is no asset liquidation ($a_1 = 0$), and $\frac{r_2}{r_1}$ when there is asset liquidation ($a_1 > 0$), for $t = 2$ relative to $t = 1$. Hence, late consumers do not withdraw at $t = 1$, $w = 0$.

Central bank The central bank policy rates can influence real activity because of the nominal rigidity of $t = 1$ deposit rates and the partial irreversibility of firm investment set at $t = 0$.

Central bank discretion is required to set policy rates according to the macro state for buffering the economy against aggregate liquidity and asset risk. Efficiency requires $R_1^m(\omega)$ and $R_2^m(\omega)$ to be set by the central bank dependent on the state. But such discretion allows for excessively high fiat inflation and lower long-term investment, output and consumption if the central bank has a short-term bias, $\beta^{cb} < \beta^{\ell}$, which provides the rationale for private digital currency analyzed in the next section. The state-dependence of the $t = 1$ realized returns $R_1^f(\omega)$ and $R_1^m(\omega)$ (and/or R_1^d having state-dependence), and the central bank setting the optimal $R_1^{m^*}(\omega)$, are both required for the optimal allocation to be implemented.

The state ω being observable but not verifiable when realized is what gives the central bank, as supplier of fiat money and hence ability to set the return on money, the role of setting the return on money (and hence return on bank loans to firms) based on observing the state. Otherwise, the return R_1^f (as well as R_1^d) could not be contingent on the state. If the state was verifiable at $t = 1$ and contractible at $t = 0$, and the central bank was passive (not setting a state-contingent $R_1^m(\omega)$), the equilibrium $R_1^f(\omega)$ would induce too low of storage at $t = 0$, $g_0 = 1 - a_0 < g_0^*$, and hence too low of early type consumption $c_1 < c_1^*$ (when consumers have a CRRA greater than one), for reasons analogous to why in Diamond-Dybvig the market provides $c_1 < c_1^*$ when there are no banks versus banks providing c_1^*.

State-dependence of returns The state-dependence of the $t = 2$ returns $R_2^f(\omega)$ and $R_2^m(\omega)$ are straightforward, since the borrowing and lending of $L_1^f(\omega)$ and $M_1^b(\omega)$ is chosen at $t = 1$ when ω is observed. The state-dependence of the $t = 1$ returns $R_1^f(\omega)$ and $R_1^m(\omega)$ are motivated and can be microfounded by considering a slight extension of the model. At $t = 0$, banks lend L_0^f and hold M_0^b for a one-half period at the non-contingent and non-state dependent rates $R_{\frac{1}{2}}^f$ and $R_{\frac{1}{2}}^m$, where period $t = \frac{1}{2}$ occurs in between periods $t = 0$ and $t = 1$, and the state ω is realized and observable at $t = \frac{1}{2}$. At $t = \frac{1}{2}$, banks and firms need to rollover the loan, $L_{\frac{1}{2}}^f = L_0^f R_{\frac{1}{2}}^f$, and banks need to rollover their fiat reserves held, $M_{\frac{1}{2}}^b = M_0^b R_{\frac{1}{2}}^m$, which can be at the returns $R_1^f(\omega)$ and $R_1^m(\omega)$, respectively, that are dependent on the state ω observed at $t = \frac{1}{2}$.

Investment technology Allowing for consumers and/or banks to have the investment technology for assets and efficient storage would not qualitatively change the results. Banks are required to provide the sharing of idiosyncratic liquidity risk between consumers who are early and late types. Public digital currency and nominal contracts are required for the central bank to implement optimal state-contingent consumption. If banks were to have the asset and storage investment technology, or if firms were able to take deposits, the results of the paper would be unchanged. However, separating the bank and firm (rather than assuming a hybrid bank/firm) gives much more insight into the distinct underlying roles of bank deposit borrowing, bank lending, and firm borrowing.

4 Private digital currency

The use of private digital currency, instead of central bank fiat money and CBDC, is now analyzed with banks taking deposits and lending in private digital currency.

4.1 Dates and issuer

At $t = 0$, private digital currency M^v is created by a private issuer. The issuer, as well as banks, consumers and firms, can hold and make transactions with the private digital currency. The optimization problems for banks, consumers and firms are the same as in the previous section 3 with private digital currency in place of fiat money

Dates In order to permit money to have a continuation value after $t = 2$, the three periods $t = 0, 1, 2$ occur within individual, non-overlapping dates τ that repeat indefinitely: $\tau = 0, 1, ..., \infty$. The analysis and results are focused within a single generic date τ for a direct comparison to the optimal allocation and fiat money results.

The initial amount M^v of private digital currency is created only once at date $\tau = 0$ and can be stored across periods and dates, while goods are perishable after period $t = 2$ of each date. Within each date τ, a new generation of consumers endowed with goods is born at $t = 0$, there is free entry of new banks and firms, and these agents live only within the date, while the issuer is infinitely-lived. The random macro state $(\tilde{\lambda}_\tau, \tilde{r}_{2,\tau})$, denoted as ω_τ is i.i.d. across dates.

Private issuer At each date τ, the issuer can use its private digital currency to buy $q^y_{0,\tau}$ goods at $t=0$. The issuer has a technology to store $g^y_{0,\tau}$ goods at $t=0$ that the issuer can sell or consume as profit $c^y_{2,\tau}$ at $t=2$. The issuer holds $M^y_{t,\tau}$ private digital currency at $t \in \{0,1,2\}$. The issuer is a competitive price-taker and maximizes expected profits over all dates:

$$\left.\begin{array}{lll}\max_{Q^y} \sum_{\tau=0}^{\infty} E[c^y_{2,\tau}] & & \\ \text{s.t.:} & t\text{=}0: & g^y_{0,\tau}P_{0,\tau} + M^y_{0,\tau} \leq M^y_{2,\tau-1} \\ & t\text{=}1: & M^y_{1,\tau} \leq M^y_{0,\tau} \quad \forall \omega_\tau \\ & t\text{=}2: & M^y_{2,\tau} \leq (g^y_{0,\tau} - c^y_{2,\tau})P_{2,\tau} + M^y_{1,\tau} \quad \forall \omega_\tau, \end{array}\right\} . \tag{45}$$

which includes the issuer's budget constraint for each period $t = 0, 1, 2$ within date τ. $M^y_{2,\tau-1}$ is the issuer's private digital currency held entering date τ from the previous date $\tau-1$, where $M^y_{2,\tau-1} \equiv M^v$ at date $\tau = 0$, and $Q^y \equiv \{g^y_{0,\tau}, M^y_{0,\tau}, \{M^y_{t,\tau}(\cdot)\}_{\omega_\tau, t\in\{1,2\}}\}_\tau$. $P_{0,\tau}$, $P_{1,\tau}(\omega_\tau)$ and $P_{2,\tau}(\omega_\tau)$ are the prices of goods, and $\Pi_{1,\tau}(\omega_\tau)$ and $\Pi_{2,\tau}(\omega_\tau)$ are inflation, in terms of private digital currency as numeraire within date τ. The subscript τ indicating the date is typically suppressed except when included to distinguish across dates.

Equilibrium The model with private digital currency is equivalent to the model with fiat reserves and CBDC in section 3 with the following modifications: i) private digital currency in place of fiat money, ii) no bank borrowing from the central bank, hence $M^b = 0$ and $M^b_1 \geq 0$, and iii) the nominal return on money is $R^m_t = R^c_t = 1$ for $t = 1, 2$. With these three modifications, banks, consumers and firms have optimization problems with budget constraints that are the same as in section 3. As in section 3, we do not consider firms holding private digital currency, as they would not in equilibrium if considered. For simplicity of the analysis, we set the late consumer's discount factor to one, $\beta^\ell = 1$, for the remainder of the paper.

Definition 2 *The definition of equilibrium is updated from Definition 1 in section 3 with i) the quantity Q^y, which is required to satisfy the issuer optimization (45), included, ii) the central bank optimization (21) excluded, and iii) market clearing for goods at $t = 0$ is replaced by $q_0 + g^y_0 = 1$ and at $t = 2$ is replaced by $c_2 + c^y_2 = q_2 + g^y_0$.*

4.2 Velocity of money

For each period $t = 0, 1, 2$, the budget constraints for banks, consumers and firms correspond to within-period repeated cycles of buying and selling goods, with a velocity of money greater than one, if the quantity of money is sufficiently small. These cycles apply i) in the fiat model of section 3 for a small quantity M^b of fiat money, and ii) in the private model of this section 4 for a small quantity M^v of private digital currency. In either case, there are also cycles of deposits within period $t = 0$ and cycles of withdrawals within periods $t = 1$ and $t = 2$.

Deposit cycles at t=0 For the fiat model, banks have initial money M^b borrowed from the central bank. For the private model, banks have initial money from deposits by consumers who initially sell a fraction of their goods to the issuer for private digital currency. For either model, banks lend their initial money to firms and/or hold it as reserves. Firms buy some of the consumer goods, and consumers deposit and/or hold the money as digital currency. Banks lend and/or hold new deposits, firms buy more goods, and consumers deposit and/or hold additional proceeds, in repeating cycles until consumers have sold all of their endowment goods.

Withdrawal cycles at t=1,2 At each period $t = 1$ and $t = 2$, consumers withdraw a part of their deposits, buy some goods from firms, and firms repay a part of their loans, in repeating cycles until all withdrawals are made and all goods are sold for the period.

Velocity and quantity of money The velocity of money holds in accordance with the quantity theory of money identity for each period $t = 0, 1, 2$ for the fiat and private digital currency models. We use the index $\iota \in \{b, v\}$, where $\iota = b$ corresponds to the fiat model and $\iota = v$ corresponds to the private digital currency model. The quantity theory in our context states that $\nu_t^\iota(M^\iota) \times M^\iota = q_t^\iota \times P_t^\iota$, where $\nu_t^\iota(M^\iota)$ is the velocity of the quantity of outside money M^ι, and q_t^ι is the quantity of goods bought (or equivalently sold) at the price level P_t^ι, for $\iota \in \{b, v\}$ and $t = 0, 1, 2$. We can also apply the quantity theory to the velocity and quantity of inside money based on D^ι. The velocity of money at $t = 0$ is $\nu_0^\iota(M^\iota) = \frac{1xP_0^\iota}{M^\iota} = \frac{P_0^\iota}{M^\iota}$ in terms of the quantity of outside money M^ι (i.e. base money, or M0), and is $\nu_0^\iota(D^\iota) = \frac{1xP_0^\iota}{D^\iota} = \frac{P_0^\iota}{D^\iota}$ in terms of the quantity of inside money D^ι (i.e. M1 - M0), for $\iota \in \{b, v\}$. The velocity of outside money M^ι at $t = 1, 2$ is $\nu_t^\iota(M^\iota) = \frac{q_t^\iota P_t^\iota}{M^\iota}$ for $\iota \in \{b, v\}$. The velocity of inside money D^ι at $t = 1$ is $\nu_1^\iota(D^\iota) = \frac{q_1^\iota P_1^\iota}{\lambda' D^\iota R_1^{d_\iota}}$ and at $t = 2$ is $\nu_2^\iota(D^\iota) = \frac{q_2^\iota P_2^\iota}{(1-\lambda') D^\iota R_2^{d_\iota}}$ for $\iota \in \{b, v\}$.

4.3 Analysis

Inflation Because there is no increase in the supply of private digital currency, there is no increase in the price level and no inflation in the long term within or across dates. This result arises in equilibrium for agents to be willing to hold and transact in the private digital currency.

Lemma 3 *There is no long-term inflation for private digital currency within and across dates:* $\frac{P_{2,\tau}(\omega_\tau)}{P_{0,\tau}} = \Pi_{1,\tau}(\omega_\tau)\Pi_{2,\tau}(\omega_\tau) = 1$, *and* $P_{0,\tau+1} = P_{0,\tau}$, *for all dates* τ *and states* ω_τ.

The private issuer's budget constraints for $t = 0, 1, 2$ at each date τ can be consolidated as a single budget constraint solved for $c_{2,\tau}^y$ as

$$c_{2,\tau}^y = g_0^y(1 - P_{0,\tau}/P_{2,\tau}) - (M_{2,\tau}^y - M_{2,\tau-1}^y)/P_{2,\tau}. \tag{46}$$

Market clearing conditions for goods implies that the issuer holds the private digital currency across dates, $M_{2,\tau}^y = M_{2,\tau-1}^y$, and no long-term inflation within a date implies $P_{2,\tau} = P_{0,\tau}$. Substituting from these two equalities into equation (46), the issuer has zero consumption profit at each date: $c_{2,\tau}^y = 0 \; \forall\tau$.

Firms, banks and consumers The firm's first-order conditions and complementary slackness for storage g_1 and liquidation a_1 at $t = 1$ are the same with private digital currency as with fiat money in section 3, based on the same condition $r_2^f = r^k$ and cutoffs $\check{\lambda}^f(r_2)$ and $\hat{\lambda}^f(r_2)$ from equations (33) and (35). The firm stores $g_1 > 0$ for $r_2^f = 1$ when $\lambda' < \check{\lambda}^f(r_2)$ and liquidates $a_1 > 0$ for $r_2^f = \frac{r_2}{r_1}$ when $\lambda' > \hat{\lambda}^f(r_2)$. Otherwise, $g_1 = a_1 = 0$ for $r_2^f = r^k \in [1, \frac{r_2}{r_1}]$ when $\lambda' \in [\check{\lambda}^f(r_2), \hat{\lambda}^f(r_2)]$, and initial investment is implicitly determined by $E[r_2^f] = \bar{r}$.

The bank's first-order condition and complementary slackness for M_t^b for $t = 1, 2$ require $R_1^f \geq R_1^m = 1$ and $R_2^f \geq R_2^m = 1$, which bind if $M_t^b > 0$ for $t = 0$ and $t = 1$, respectively. These result in the same consolidated bank budget constraint, equation (15), as in section 3. R_1^d and R_1^f are each constants not contingent on the realized state ω. In the main analysis on runs and efficiency, we consider $R_1^d = R_1^f$, which implies from equation (15) that the implicit return on holding deposits equals the nominal loan return between $t = 1$ and $t = 2$,

$$\frac{R_2^d(\omega)}{R_1^d} = R_2^f(\omega) \quad \forall \omega, \tag{47}$$

and from equations (8) and (9) that $\check{\lambda}^f(r_2) = \check{\lambda}(r_2) = \frac{g_0}{g_0 + a_0 r_2}$ and $\hat{\lambda}^f(r_2) = \frac{g_0}{g_0 + a_0 r_1}$.

Consumers choose whether to hold private digital currency according to the same conditions as in lemma 2, section (3), where $R_1^c = R_1^c = 1$ for private digital currency. Hence, consumers hold deposits rather than private digital currency at $t = 0$ if $R_1^d \geq 1$ and at $t = 1$ if $R_2^d(\omega) \geq R_1^d$. Following section 3, a late consumer withdrawing early to buy goods at $t = 1$, $w(\cdot) = 1$, requires $c_1^\ell \geq c_2^\ell$, i.e. $\frac{R_1^d}{P_1(\cdot)} \geq \frac{R_2^d(\cdot)}{P_2(\cdot)}$ and equivalently $\frac{R_2^d(\cdot)}{R_1^d} \leq \Pi_2(\cdot)$, since $\beta^\ell = 1$ implies that $r^\ell \equiv \beta^\gamma = 1$.

Lemma 4 *Late consumers do not withdraw early to buy goods at* $t = 1$ *if* $\frac{R_2^d(\cdot)}{R_1^d} > \Pi_2(\cdot)$.

We consider $R_1^d = R_1^f \geq 1$ in the main analysis, such that consumers do not hold private digital currency at $t = 0$, $M_0^c = 0$. We discuss the $t = 1$ deposit rate R_1^d and loans rate R_1^f more generally at the end of the section.

Quantity of money For simplicity of the presentation and results, we proceed by considering the quantity of outside money going to zero in the limit. With $M^v \to 0$ for private digital currency, as equivalent to $M^b \to 0$ for fiat money, the velocity of outside money at each period $t = 0, 1, 2$ goes to infinity in the limit, $\nu_t^\iota(M^\iota) \to \infty$, for $\iota \in \{v, b\}$. The private model in this section, and the fiat model in section 3, are unchanged. For each model, money is maintained as means of exchange for transactions at each period $t = 0, 1, 2$, and as the unit of account for nominal quantities, prices and returns. The consideration of banks and consumers holding money is maintained because the analysis is based on a representative bank and consumer, which is equivalent to one bank and one consumer each within a continuum of banks and consumers, respectively, holding money at the margin.

For the fiat model, all of the results and analysis are unchanged, including the equilibrium real returns and quantities and nominal returns, prices and quantities, except that market clearing implies that $M^b = M_t^b = M_t^c$ for $t = 0, 1$. For the private model in this section, lemma 3 above is unchanged. The remaining analysis and results in this section for digital currency runs and efficiency are not qualitatively different from considering instead a positive amount of private digital currency $M^v > 0$.

4.4 Private digital currency runs

We first show there are no bank runs caused by the threat of late consumers withdrawing deposits early to buy goods at $t = 1$, which is analogous to bank runs in traditional real-economy based banking models. For this threat, we consider late consumers $i \in [\lambda, 1]$ who may withdraw early, $w^i = 1$, to buy goods at $t = 1$ but not to hold private digital currency at $t = 1$. Hence, $c_1^{\ell i} = \frac{w^i D R_1^d}{P_1}$, $M_1^{ci} = 0$ and $c_2^{\ell i} = \frac{(1-w^i) D R_2^d}{P_2}$.

Deflation and negative interest rates As in the last section, when there are relatively few early consumers and low asset returns, $\lambda < \check{\lambda}(r_2)$, the real lending rate must equal one ($r_2^f = 1$) for firms to optimally store goods $g_1 > 0$ at $t = 1$, which implements optimal consumption for late and early types that with $\beta^\ell = 1$ is equal, $c_2 = c_1$. For fiat money, there is sufficiently high fiat inflation Π_2 at $t = 2$ (whether optimal or excessively high) through a low enough price level P_1 and high enough price level P_2 to implement the low real rate $r_2^f = 1$.

For private digital currency, when $\lambda < \check{\lambda}(r_2)$ and there is a real asset return loss $r_2 < 1$, which we denote as $\lambda < \check{\lambda}(r_2 < 1)$, sufficiently high inflation at $t = 2$ to implement $c_2 = c_1$ cannot occur as in the fiat model. Instead, the optimal real rate and equal consumption could be implemented through deflation with negative nominal interest rates on loans and deposits between $t = 1$ and $t = 2$. For $\lambda < \check{\lambda}(r_2 < 1)$, optimal consumption is less than one: $c_1 = c_2 = g_0 + a_0 r_2 < g_0 + a_0 = 1$.

Consumers holding deposits have the budget constraint $D = P_0$ at $t = 0$. Consumption for early types can be expressed as the real return on deposits at $t = 1$: $c_1 = \frac{D R_1^d}{P_1} = \frac{R_1^d}{\Pi_1}$. Since $c_1 < 1$, and consumers require $R_1^d \geq 1$, there is short-term inflation at $t = 1$: $\Pi_1 > R_1^d \geq 1$. No long-term inflation, $\Pi_1 \Pi_2 = 1$, thus requires short-term deflation at $t = 2$: $\Pi_2 < 1$.

The optimal storage of goods at $t = 1$, $g_1 > 0$, requires that the short-term real rate equals one, $r_2^f = \frac{R_2^f}{\Pi_2} = 1$, and hence a negative nominal interest rate on loans rolled over at $t = 1$, $R_2^f = \Pi_2 < 1$. The nominal return on loans below one, $R_2^f < 1$, only permits the bank to pay a return on deposits that is lower at $t = 2$ than at $t = 1$: $R_2^d = R_2^f R_1^d < R_1^d$. The implicit interest rate on holding deposits between $t = 1$ and $t = 2$ is negative: $\frac{R_2^d}{R_1^d} = R_2^f < 1$.

Thus, deflation with negative nominal interest rates at $t = 2$ implements firms storing goods at $t = 1$ and optimal equal consumption for depositors withdrawing and buying goods at $t = 2$ as at $t = 1$. Note that since late consumers are indifferent, an amount up to $\lambda' - \lambda$ late consumers may withdraw and buy goods at $t = 1$, where $\lambda' \leq \check{\lambda}(r_2 < 1)$. Firms simply store fewer goods at $t = 1$, and consumption for early and late types is unchanged. In contrast, a fraction $\lambda' > \check{\lambda}(r_2 < 1)$ of withdrawals at $t = 1$ is not an equilibrium, as consumption would be strictly greater from withdrawing at $t = 2$ than $t = 1$: $c_2^\ell > c_1^\ell$.

Lemma 5 *If late consumers do not withdraw at* $t = 1$ *to hold private digital currency, then the incentive constraint for late consumers holds,* $c_2 \geq c$*, and there are no bank runs.*

Digital currency runs We now examine late consumers who may withdraw early in an attempt to hold private digital currency at $t = 1$ for buying goods at $t = 2$. Late consumer $i \in [\lambda, 1]$ withdraws $w^i \in \{0, 1\}$ to either buy goods at $t = 1$ or hold $M_1^{ci} = w^i D R_1^d$. We show that for $\lambda < \check{\lambda}(r_2 < 1)$, when $\lambda < \hat{\lambda}(r_2)$ and $r_2 < 1$, a digital currency run is triggered because of the deflation at $t = 2$.

First consider the partial equilibrium for macro states with $\lambda < \hat{\lambda}(r_2 < 1)$ holding fixed the the deposit returns and prices from lemma 5. A late consumer would attempt to withdraw at $t = 1$, $w = 1$, for the higher nominal return $R_1^d > R_2^d$, and hold the private digital currency, $M_1^c = D R_1^d$, to buy goods at the lower price level at $t = 2$ with deflation $\Pi_2 < 1$. This would give greater consumption, $c_2 = \frac{D R_1^d}{P_2} > D$, than by withdrawing and buying goods at $t = 1$, $c_1^\ell = \frac{D R_1^d}{P_1} < D$, or at $t = 2$, $c_2 = \frac{D R_2^d}{P_2} < D$.

With all late consumers attempting to withdraw early and hold private digital currency, the withdrawal fraction is $\lambda' = 1$. However, the bank cannot pay the full withdrawals, and the bank and firm both default at $t = 1$. To analyze defaults, we define the actual realized return paid as $R_1^j(\omega) \equiv \delta_1^j(\omega) \hat{R}_1^j$ on bank deposits, for $j = d$, and on loans, for $j = f$, where $\delta_1^j(\omega) \leq 1$ is the fraction paid on the return

$\hat{R}_1^j$ that is initially contracted at $t = 0$. A default is signified by $\delta_1^j(\cdot) < 1$, which requires the borrowing bank or firm to pay all revenues possible to maximize $\delta_1^j(\cdot)$. For a bank default, there is a pro rata sharing rule among all withdrawals at $t = 1$. The contracted return $\hat{R}_1^j$ is not state-contingent. The realized return is not state-contingent for no default, $R_1^f = \hat{R}_1^j$ for $\delta_1^j(\cdot) = 1$, and is only state-contingent for a default, $R_1^f(\omega) = \delta_1^j(\omega)\hat{R}_1^j < \hat{R}_1^j$ for $\delta_1^j(\omega) < 1$.

In equilibrium, for $\lambda' = 1$, the bank defaults and cannot rollover any loans to firms, $L_1^f = 0$, in order to maximize withdrawal payments at $t = 1$. Without loan rollovers, firms default at $t = 1$ and have to completely liquidate their assets, $a_1 = a_0$, to maximize loan repayments at $t = 1$.

At $t = 1$, the budget constraint for the bank is $D\delta_1^d\hat{R}_1^d = L_0^f\delta_1^f\hat{R}_1^f$ and for the firm is $L_0^f\delta_1^f\hat{R}_1^f = q_1 P_1$, where $P_1 = \dfrac{\lambda_1 D\delta_1^d\hat{R}_1^d}{q_1}$. By substituting and simplifying among these, consumption is lower than in the case of no run, with

$$c_2 = c_1 = \delta_1^d = g_0 + a_0 r_1 < g_0 + a_0 r_2 < 1. \tag{48}$$

The bank's $t = 0$ loans to firms are too illiquid to receive their par value repaid back at $t = 1$ to pay out withdrawals at $t = 1$. At $t = 2$, the bank has no revenues and $R_2^d = 0$. Late consumers fully withdraw $w = 1$ and receive pro rata with early consumers an amount based on the the default fraction $\delta_1^d < 1$.

Proposition 5 *When there is a loss on the asset $r_2 < 1$ and a large fraction of late consumers $1 - \lambda > 1 - \check{\lambda}(r_2)$, there is a digital currency run in the form of digital currency withdrawal attempts by late consumers that force a complete asset liquidation by the bank and firms.*

This proposition shows that for $\lambda < \check{\lambda}(r_2 < 1)$, inefficient digital currency runs occur resulting in $c_2 = c_1 = g_0 + a_0 r_1 << 1$ despite that if there were no run, the late consumer incentive constraint would hold with equal consumption for late and early consumers, and at a higher amount of consumption with no liquidation, $c_2 = c_1 = g_0 + a_0 r_2$, shown by lemma 5.

As a stark illustration, consider $\lambda < \check{\lambda}(r_2 < 1)$ with only a very slight asset loss, where $r_2 = 1 - \varepsilon$ for an arbitrarily small $\varepsilon > 0$. Without digital currency as shown above, a bank efficiently survives without a run or any liquidation. But with a private digital currency, the slight loss is magnified by a complete run and liquidation even as the bank insolvency and asset loss ε approaches zero in the limit.

Otherwise no runs When $\lambda < \check{\lambda}(r_2)$ and there is not an asset loss, $r_2 \geq 1$, the real rate $r_2^f = 1$ required for firms to store $g_1 > 0$ at a MRT of $r^k = 1$ does not require deflation or negative nominal interest rates. There are no digital currency runs, and early and late types have optimal equal consumption. Since $\frac{R_2^f}{\Pi_2} = r_2^f = 1$ and $\frac{R_2^d}{R_1^f} = R_2^f$, inflation and the nominal rates at $t = 2$ are $\Pi_2 = R_2^f \geq 1$ and $R_2^d \geq R_1^d$, and consumption is $c_1 = c_2 = g_0 + a_0 r_2 \geq 1$, with these three inequalities binding if $r_2 = 1$ and not binding if $r_2 > 1$.

When $\lambda > \check{\lambda}(r_2)$, late consumers have $R_2^d > R_1^d$ and $c_2^\ell > c_1$ without needing firms to store goods, $g_1 = 0$. Even if there is a loss on asset returns, $r_2 < 1$, there are not too many late types needing to consume out of the low asset returns at $t = 2$.

Corollary 2 *When $\lambda < \hat{\lambda}(r_2)$ and $r_2 > 1$, or when $\lambda \geq \check{\lambda}(r_2)$ for all r_2, there are no digital currency runs.*

4.5 Efficiency

Optimal consumption cannot be implemented even when digital currency runs do not occur, since R_1^d and R_1^f are each constants not contingent on the realized state. In particular, ex-interim efficiency requires that equation (43) holds, which since $R_1^f = R_1^m$ and $\beta^\ell = 1$ is written as:

$$\frac{R_1^f}{R_1^d} = \lambda' + (1 - \lambda')/(r^k(\omega))^{1-\gamma}. \tag{49}$$

Relative consumption is $\frac{c_2(\cdot)}{c_1(\cdot)} = r^k(\cdot)$ from equation (41), and $R_2^f(\cdot) = R_2^d(\cdot)$ from equation (15).

Hence, ex-interim efficiency requires $\frac{R_1^f}{R_1^d} = 1$ for $r^k(\cdot) = 1$ when $\lambda \leq \check{\lambda}(r_2)$, but requires $\frac{R_1^f}{R_1^d} > 1$ for $r^k(\cdot) > 1$ when $\lambda > \check{\lambda}(r_2)$ if the CRRA is greater than one (i.e. $\gamma < 1$). With $\frac{R_1^f}{R_1^d} = 1$, there is efficient storage at $t = 1$ and equal consumption $c_2 = c_1$ when $\lambda' < \check{\lambda}(r_2)$ and $r_2 \geq 1$, since $\check{\lambda}^f(r_2) = \check{\lambda}^f(r_2) \equiv \frac{g_0}{g_0 + a_0 r_2}$. But there is inefficiently low liquidation at $t = 1$ and low consumption for early types when $\lambda > \hat{\lambda}(r_2)$, since $\hat{\lambda}^f(r_2) = \frac{g_0}{g_0 + a_0 r_1} \geq \hat{\lambda}(r_2) \equiv \frac{g_0}{g_0 + a_0 r_2/(r_2/r_1)^\gamma}$, which is binding only for $\gamma = 1$.

Proposition 6 *Ex-interim continuing investment, output, consumption, and risk-sharing between early and late consumers is generally inefficient, with inefficiently lower than optimal ex-interim liquidation for risk-sharing for $\lambda > \hat{\lambda}(r_2)$.*

Lower initial investment Because there is an inefficient complete asset liquidation $a_1 = a_0$ in a digital currency run, for the states when $\lambda < \check{\lambda}(r_2 < 1)$, there is a lower amount of initial asset investment than otherwise optimal. The initial assets, $a_0 < a_0^*$, are implicitly defined by the firm's first order condition, $E[r_2^f] = \bar{r}$ if $a_0 > 0$. However, the lower initial asset investment does not eliminate digital currency runs, unless they would occur frequently enough and liquidation costs are high enough (reflected by a low r_1) such that $E[r_2^f] = \bar{r}$ does not hold for $a_0 > 0$. For this case, there would be no asset investment, $a_0 = 0$, and we would have $R_2^d(\omega) = R_1^d = 1$ and $c_2(\omega) = c_1(\omega) = 1$ for all states ω.

Initial deposit and loan rates We do not analyze the optimal R_1^f and R_1^d, as the results would not be qualitatively changed by considering $R_1^f > 1$ or $R_1^d \gtrless 1$. For $R_1^f > 1$, the digital currency run results would not change, and the ex-interim efficiency results would only be affected in a qualitative way based on R_1^f as relative to R_1^d in the ratio $\frac{R_1^d}{R_1^f}$, as according to equation (49).

For $R_1^d > 1$, there would be an increase in risk-sharing among early and late consumers, as according to equation (49), when runs do not occur. However, digital currency runs would occur more frequently, since $\check{\lambda}^f(r_2)$ increases in R_1^d, and hence the probability of the states with $\lambda < \check{\lambda}(r_2 < 1)$ increases in R_1^d as well. For $R_1^d < 1$, the converse would hold, with less frequent runs but a decrease in risk-sharing among early and late types when runs do not occur. However, $R_1^d < 1$ could not hold in equilibrium in a more generalized setting that allowed late consumers at $t = 1$ to make an initial bank deposit for a return of $\frac{R_1^d}{R_1^d}$ at $t = 2$, since consumers would then only hold goods and no deposits at $t = 0$.

4.6 Discussion

Money The model is developed to allow for a parsimonious representation of outside money in the form of traditional fiat reserves, CBDC, and private digital currency. The interpretation is that the central bank and private issuer each have a technology to create a mutually distinguishable currency. Each currency is a technology which agents can use to make verifiable payment transactions without double-spending. The distinction of the two currencies is that the central bank technology allows for the central bank to create new quantities of money at any period. This distinction is motivated by the fact that a central bank is ultimately controlled by a sovereign government, which does not have a commitment technology to limit its fiat ability to issue money. The private issuer technology allows the issuer to commit to create a single fixed quantity of private digital currency only once, at the initial period and date. This distinction is motivated by blockchain technology, distributed ledgers, and protocols such as used with bitcoin, which has shown a strong ability to commit to issuing a fixed quantity of a specific private digital currency over a pre-specified time period.

The distinction between digital currency, whether CBDC or private digital currency, and traditional fiat reserves, is that digital currency enables individuals and firms to make transactions without requiring banks for payments. Without digital currency, bank payment systems are typically required, rather than just using paper currency for example, for the very large size and scale of economic and financial transactions in the economy.

Transaction costs In order to focus on the potential future role for digital currency to compete with bank payments as an efficient means of payment, we make the simplifying assumption of no transactions costs for using either digital currency or bank deposits.[21] This precludes the channel for digital currency to have a positive value purely from a direct means-of-payment liquidity premium. A liquidity premium value for an outside money would be equal to the present value of future payment liquidity services for non-instantaneous transactions, as in Bias et al. (2018b). More efficient payments imply a lower liquidity premium value for outside money. With the simplification of assuming costless and effectively instant bank and digital currency payments, there is no liquidity premium value.

In practice, private digital currency utilizes a decentralized distributed ledger with blockchain technology and requires a protocol to achieve consensus for payments

[21] Payments Canada et al. (2018) demonstrate the potential for widespread banking payments without reliance on a central bank that would be required for banks to take private digital currency deposits. They describe the development and testing in Canada for efficiently settling large-scale wholesale interbank payments with distributed ledger technology. A "notary node" consensus model shows promise for settlement finality, which is required but not achieved with a proof-of-work protocol. Parlour et al. (2017) show that fintech innovation in the bank payment system can reduce banks' need for intermediate liquidity in the interbank market, which results in an increase in bank lending and productive efficiency.

transactions in such a 'trustless' environment.[22] Most public digital currency payments under current consideration would likely utilize a 'trusted' centralized central bank ledger.[23] With developments in methods for private digital currency payments to support consensus for transactions in a more cost effective manner, such as with proof-of-stake rather than proof-of-work protocols,[24] or with second-layer protocols such as the Lightning Network to increase scalability,[25] private digital currency has the potential to be used as an efficient means of digital payments similar to or even more advanced than electronic payments that are cleared and settled within the banking system.

5 Consumer lending

We now consider consumers directly lending to firms, without banks as intermediaries, using either private digital currency or CBDC. We then analyze whether consumer lending or bank lending, using private digital currency or fiat money and CBDC, is the overall equilibrium outcome for the economy.

Consumer lending private digital currency We first show that consumer lending with private digital currency has the trade-off of not providing the maturity and risk transformation of bank lending but does avoid digital currency runs. The model is equivalent to that for private digital currency with banks in section 4.1, with the modification that deposits are replaced by direct loans from consumers to firms.

[22] Kroll et al. (2013) examine bitcoin as a consensus game using costly computational mining as proof-of-work for transaction consensus, and which also requires a separate governing consensus for the rules of the bitcoin protocol. Biais et al. (2018a) show that bitcoin transaction consensus using the mining proof-of-work protocol is a Markov perfect equilibrium but that consensus over the protocol is a coordination game with multiple equilibria. Cong et al. (2018) examine methods for moderating the natural concentration of mining pools, and Easley et al. (2017) explain market-based transaction fees charged in addition to mining rewards.

[23] Raskin and Yermack (2018) describe how central bank digital currency (CBDC) would enable households to hold such public digital currency directly in accounts at the central bank instead of in deposit accounts at commercial banks.

[24] Saleh (2018a,b) shows that protocols such as proof-of-stake or proof-of-burn can overcome the large computing resources costs required for proof-of-work consensus protocols, such as for bitcoin, which Parham (2017) demonstrates are prohibitive on a large scale.

[25] Poon and Dryja (2016) describe how the Lightning Network, which has reached increasing success in recent small-value tests, acts as a decentralized network off of the bitcoin blockchain for micropayments in bitcoin, with net payments then transacted on the bitcoin blockchain.

At $t = 0$, consumers sell a part of their endowment goods to the issuer for private digital currency that consumers hold or lend to firms, which buy goods from consumers who further hold or lend in continuing cycles until consumers have sold all of their goods. Hence, the velocity of outside money is equivalent to that for private digital currency with banks.

The consumer's optimization (19) is updated with D, R_1^d and $R_2^d(\omega)$ replaced by L_0^f, R_1^f and $\frac{R_2^f(\omega)}{R_1^f}$, respectively. The firm's optimization does not change. We interpret this as a firm only able to borrow from a single consumer at $t = 1$ and to rollover the loan from that consumer at $t = 1$. This is equivalent to the firm borrowing and rolling over from a single bank in the previous sections. With bank lending, this restriction on firms has no impact since all banks are identical in each of the three periods. With consumer lending, this restriction distinguishes that there is not the maturity transformation that banks provide.

Definition 3 *The definition of equilibrium is updated from Definition 2 in section 4.1 with the bank quantity Q^b and bank optimization (13) excluded.*

At $t = 1$, early consumers receive the return R_1^f on their $t = 0$ loans to firms. Since early consumers do not roll over any loans to their firms, these firms have to liquidate their assets, $a_1 = a_0$. These firms sell their $q_1 = g_0 + a_0 r_1$ goods, and their budget constraint at $t = 1$ is $L_0 R_1^f = q_1 P_1$. Hence, early types have consumption $c_1 = \frac{L_0 R_t^f}{P_1} = q_1 \leq 1$, with $c_1 < 1$ if $a_0 > 0$.

Late consumers roll over the full amount of their loans at $t = 1$ and receive the compounded return $R_1^f R_2^f$ at $t = 2$. Their firms do not liquidate any of their assets and continue to store all of their inital stored goods at $t = 1$, $g_1 = g_0$. Their budget constraint consolidated over $t = 1$ and $t = 2$ iis $L_0 R_1^f R_2^f = q_2 P_2$, where $q_2 = g_0 + a_0 r_2$. Hence, late types have consumption $c_2 = \frac{L_0 R_1^f R_2^f}{P_2} = q_2$.

The liquidation by firms borrowing from early consumers results in overall inefficient underinvestment in assets by firms in terms of both continued investment at $t = 1$ and initial investment at $t = 0$.

Proposition 7 *Consumer lending directly to firms without bank intermediation avoids digital currency runs but otherwise provides for lower asset investment, overall output, efficiency, expected consumption, and risk-sharing between early and late consumers.*

Discussion Direct consumer lending relative to bank lending has a shorter expected maturity of loans, and no maturity transformation of more long-term loans that can support more firm investment in long-term illiquid assets. Bank lending provides asset returns for late consumers, as well as greater consumption for early consumers in the form of idiosyncratic liquidity shock risk insurance, provided that runs do not occur.

Under consumer lending, lower consumption for early consumers reflects that they can only lend short-term to firms. Lower consumption for late consumers reflects that even though loans end up extended for the longer term, a firm ex-ante invests less in assets than optimal, $a_0 < a_0^*$, because the firm has the ex-ante risk of having to fully liquidate its assets if its lender is realized as an early consumer ex-interim at $t = 1$. The asset underinvestment at $t = 0$ is ex-interim inefficient for the late consumer and the firm. However, the asset underinvestment is ex-ante constrained efficient for consumers and the firm because it decreases the ex-interim inefficiency of the amount liquidated at $t = 1$ for the early consumer and firm. But because of digital currency runs for banks with private digital currency deposits, direct consumer lending may be preferred.

Holding private digital currency Rather than depositing or directly lending private digital currency at $t = 0$, a marginal consumer who instead stores private digital currency receives a nominal return of one. This return and the expected consumption from it is lower than the expected return and expected consumption from either depositing or directly lending private digital currency in both risk-neutral and risk-adjusted terms, which reflects that storing private digital currency does not provide credit in the form of any type of lending to firms to enable investment.

Corollary 3 *With private digital currency, consumers either lend it to firms or hold it in the form of bank deposits rather than hold it directly.*

Consumer lending CBDC As with private digital currency, in the model with public digital currency, CBDC provides the opportunity for consumers to lend it directly to firms. If the central bank has a short-term bias, the impact of excessive fiat inflation at $t = 2$ affects the real value of fiat money in an equivalent manner regardless of whether consumers lend to firms, hold CBDC, or deposit at banks. If a consumer lends CBDC to firms, the long term amount of credit to firms is lower than as provided by banks, resulting in lower initial and continuing asset investment and lower expected consumption, as in the case of private digital currency. Hence, regardless of whether or not the central bank has a short-term bias, consumers prefer to hold fiat money as bank deposits rather than lend or hold CBDC directly.

Lemma 6 *With fiat money, consumers hold it in the form of bank deposits rather than lend CBDC to firms or hold CBDC directly, regardless of central bank fiat inflation.*

5.1 Monetary and lending equilibrium for the economy

We consider the economy with the central bank offering fiat reserves and CBDC, and a private issuer that creates private digital currency, to analyze which currency is used and whether consumers deposit at banks or lend to firms.

Whether fiat money or private digital currency is used, and whether it is lent to firms by consumers directly or through consumers depositing at banks, is determined by consumers who choose among these options. Their choice depends on the relative benefits and costs from high fiat inflation, bank maturity and risk transformation, and the probability and losses of private digital currency runs. Since consumers are ex-ante identical at $t = 0$ when they sell their endowment and choose their portfolio among the options of depositing, direct lending, and holding fiat money and private digital currency, the equilibrium for the economy can be analyzed based on the type of money and lending that maximizes consumers' expected utility. Holding digital currency is always dominated by either banking or direct lending, and direct lending is dominated by banking for the case of fiat money. Hence, the type of equilibrium for an economy is either fiat money with bank lending, or private digital currency with either bank lending or consumer lending.

Private digital currency avoids the distortion of excessive fiat inflation. However, if it is held directly, it loses the value creation from bank maturity and risk transformation. If held in the form of deposits, it creates the risk of digital currency runs. A short-term central bank that creates high enough excessive inflation is not able to use public digital currency to compete with private digital currency. Private digital currency does not discipline a central bank with a short-term bias, because the central bank faces an inherent time-inconsistency problem. A short-term central bank would not be credible if it tried to ex-ante pledge at $t = 0$ not to set low policy rates on reserves at $t = 1$ that cause excessive long-term inflation. Likewise, a short-term central bank cannot constrain itself by issuing CBDC or pledging at $t = 0$ to set sufficiently high policy rates on CBDC at $t = 1$. If the central bank has a significant enough short-term bias, reflected by a low enough β^{cb}, fiat money and public digital currency is not held and is driven out by private digital currency.

The efficiency of equilibrium investment and expected consumption in the economy can be compared relative to the optimal allocation for the real economy based on the asset liquidation return r_1, the joint distribution of the random macro state $(\tilde{\lambda}, \tilde{r}_t)$, and consumers' risk aversion and intertemporal substitution preference over consumption expressed by the CRRA on their utility function $u(c_t)$ given by $1/\gamma$.

This efficiency for consumers weakly decreases with higher fiat inflation by a central bank with discount factor $\beta^{cb} < 1$, higher insolvency risk for banks, and asset liquidation cost, and it increases with the value of bank maturity transformation. For $\beta^{cb} < 1$, fiat inflation decreases in β^{cb}. The insolvency risk for banks decreases in the probability $\rho \equiv \Pr(r_2 \geq 1)$. Asset liquidation cost decreases in r_1. The value of maturity transformation increases in $\bar{\lambda}$ and $\bar{r}_2$.

The determination of the equilibrium type for an economy can be characterized by these parameters of the economy, β^{cb}, ρ, r_1, $\bar{\lambda}$, $\bar{r}$, each relative to corresponding endogenous cutoffs $\hat{\beta}^{cb}(\cdot)$, $\hat{\rho}(\cdot)$, $\hat{r}_1(\cdot)$, $\dot{\lambda}(\cdot)$, and $\dot{r}_2(\cdot)$, respectively, where each cutoff is a function of the other parameters of the economy.

Proposition 8 *The equilibrium for an economy is:*
i) Fiat money with bank lending if $\beta^{cb} \geq \hat{\beta}^{cb}(\cdot)$ *or* $\bar{\lambda} \geq \dot{\lambda}(\cdot)$*;*
ii) Private digital currency with bank lending if $\rho \geq \hat{\rho}(\cdot)$ *or* $\bar{r}_2 > \dot{r}(\cdot)$*;*
iii) Private digital currency with consumer lending if $r_1 \geq \hat{r}_1(\cdot)$*.*

6 Hard currency

Private digital currency can be distinguished in contrast to hard currencies that are traditionally used to avoid excessive fiat inflation. While private digital currency does not allow for inflation to prevent digital currency runs on banks with losses on loans for assets, liquidity-based runs do not occur for fundamentally solvent banks. For $r_2 \geq 1$, there are no runs for any level of liquidity demand, since the price level in the economy is partially elastic relative to the macroeconomy's liquidity demand reflected by λ.

In comparison, a hard currency such as based on a gold standard has been historically used to preclude fiat inflation by creating a fixed value for the currency. A hard currency avoids fiat inflation but has a fixed value relative to the economy's macro shocks, which exacerbates recessions and creates a fragile banking system that is more susceptible to bank runs.

6.1 Real model

Consider a hard currency used instead of fiat money or a private digital currency. The hard currency has a fixed value in terms of goods, which creates a constant price level of goods at each period. The price level can be normalized to one, $P_t = 1$, for each period $t = 0, 1, 2$, and hence inflation is $\Pi_t = 1$ at $t = 1, 2$.

With a fixed nominal price level, the model of the economy in section 3 is equivalent to that of a real model. Goods replace money for transactions and as numeraire for deposit and loan contracts, with $M_t^b = M_t^c = 0$ for $t = 0, 1, 2$ and $R_t^m = R_t^c = 1$ for $t = 1, 2$. Consumers deposit their goods at banks to lend to firms at $t = 0$. Firms repay loans and banks pay withdrawals in goods at $t = 1, 2$. Hence, any early withdrawals by late consumers are paid in terms of goods at $t = 1$, so there is no distinguishing of bank runs in terms of currency versus for obtaining goods at $t = 1$. Uppercase and lowercase notation are equivalent since nominal and real are equivalent for hard currency values. For example, R_1^d is equivalent to a real value of goods. With this, the optimization (13) for banks, (19) for consumers and (20) for firms are in real terms for quantities, prices and returns.

Definition 4 *The definition of equilibrium is updated from Definition 2 in section 4 with the market clearing for goods, and the issuer quantity Q^y and optimization (45), excluded. The market clearing for goods in Definition 1 of section 3 now represents the budget constraints for the bank, consumer and firm consolidated for each period.*

Firms, banks and consumers The bank's and firm's budget constraints require firms to sell $q_1 = \lambda' R_1^d$ goods at $t = 1$ and remaining goods g_2 at $t = 2$, where again R_1^d is a constant not contingent on the realized state ω. The bank's and firm's budget constraints can be consolidate each each period $t = 1, 2$ as:

$$\begin{aligned} t = 1: &\quad \lambda' D R_1 \leq q_1 = g_0 + a_1 r_1 - g_1 \\ t = 2: &\quad (1 - \lambda') D R_2 \leq q_1 = (a_0 - a_1) r_2 + g. \end{aligned} \tag{50}$$

There is a single cutoff $\vec{\lambda} \equiv \frac{g_0}{DR_1^d}$, which does not depend on r_2, for storage and liquidation at $t = 1$. Firms store goods $g_1 = \lambda' D R_1^d - g_0 > 0$ if $\lambda' < \vec{\lambda}$, and firms liquidate assets $a_1 = \frac{\lambda' D R_1^d - g_0}{r_1} > 0$ if $\lambda' > \vec{\lambda}$. Storage and liquidation are generally inefficient, since one of them always occurs for each state ω except when $\lambda = \vec{\lambda}$. Following lemma 2, consumers hold deposits rather than goods at $t = 0$ if $R_1^d \geq 1$, and at $t = 1$ if $R_2^d(\cdot) \geq R_1^d$, which reflects the late consumer's incentive constraint not to run on the bank since $c_2^\ell = DR_2^d(\cdot) \geq c_1 = DR_1^d$. We consider $\hat{R}_1^d = 1$ such that consumers do not hold goods at $t = 0$, $M_0^c = 0$, and hence $D = P_0 = 1$. We discuss this deposit rate more generally at the end of the section and show that a higher rate $\hat{R}_1^d$ increases bank runs.

6.2 Bank runs

Insolvency runs Whenever there is a loss on assets, $r_2 < 1$, .insolvency-based bank runs occur for all realizations of $\lambda \in [0, 1]$. The minimum R_2^d for late consumers not to withdraw at $t = 1$ is $R_2^d = R_1^d$, which applying along with $D = 1$ to the sum of the two budget constraints in equation (50), gives $R_2^d = R_1^d \leq g_0 + a_0 r_2 < 1$ for $r_2 < 1$, and regardless of $\lambda \in [0, 1]$. With the contracted return $\hat{R}_1^d = 1$, there is a bank default at $t = 1$ with $R_1^d = \delta_1^d \hat{R}_1^d$ and $\delta_1^d < 1$. All late consumers withdraw early, $\lambda' = 1$, the bank does not rollover any loans to firms, and firms fully liquidate $a_1 = a_0$. Consumption is $c_2 = c_1 = \delta_1^d = g_0 + a_0 r_1 < g_0 + a_0 r_2 < 1$, which is the same as in equation (48) for private digital currency runs.

In contrast, however, with private digital currency, digital currency runs occur for $r_2 < 1$ only when there is also a low level of early consumers, $\lambda \leq \check{\lambda}(r_2 < 1)$. When $\lambda > \check{\lambda}(r_2 < 1)$, there is a moderate increase in the price level P_1 at $t = 1$ to lower the real consumption from buying goods with withdrawals at $t = 1$ below that at $t = 2$:

$c_2^\ell = \frac{R_2^d}{P_2} \geq c_1^\ell = \frac{R_1^d}{P_1}$.

Exess liquidation When the bank is fundamentally solvent, with $r_2 \geq 1$, there is excess liquidation when there is a moderate realization of early consumers $\lambda > \vec{\lambda}$. Banks can only roll over a limited amount of loans to firms. A marginal increase in λ requires the bank to increase its marginal withdrawal payout at a constant real amount $R_1^d = 1$, since the price level P_1 does not increase with an increase in λ. To repay the greater amount of loans not rolled over at $t = 1$, firms are forced to liquidate $a_1 = \frac{\lambda - g_0}{r_1} > 0$ of assets. The liquidation is in excess of the optimal amount (if any) of liquidation, $a_1 > a_1^*$, and consumption for late consumers is below the optimal allocation, $c_2 < c_2^*$.

Liquidity runs Liquidity-based bank runs occur when there is a high realization of early consumers, $\lambda > \widehat{\hat{\lambda}}(r_2 \geq 1) \equiv \dfrac{(g_0 + a_0 r_1) r_2 - r_1}{r_2 - r_1} \in [\vec{\lambda}, 1)$, even while the bank is fundamentally solvent for any $r_2 \geq [1, r^{\max})$.

If only early consumers $\lambda > \widehat{\hat{\lambda}}(\cdot)$ were to withdraw at $t = 1$, the asset liquidation required to pay them without defaulting is so large that banks could only pay late consumer withdrawals at $t = 2$ a lower amount than at $t = 1$: $R_2^d < R_1^d = 1$, reflecting the substantial shortage of remaining assets at $t = 2$.

A marginal withdrawal by a late consumer at $t = 1$ further decreases bank lending, and increases firm liquidation for a marginal return of less than one, $r_1 < 1$, that firms can pay to the bank for paying out the marginal withdrawal of one, $c_1^\ell = R_1^d$, required to not default at $t = 1$, which reduces the continuing investment that firms sell to pay the bank for withdrawals at $t = 2$. The marginal increase in liquidation decreases asset returns that can be paid out at $t = 2$ by $r_2 \geq 1$, which is more than the decrease in the marginal withdrawal return paid at $t = 2$ of $R_2^d < R_1^d$

Liquidating assets to pay out more late consumer withdrawals at $t = 1$ creates a downward spiral in the amount that firms and hence the bank can pay at $t = 1$. There is a complete bank run by late consumers, with $w = 1$ and $\lambda' = 1$. The bank and firms have a default and complete liquidation at $t = 1$, with $\delta_1^d < 1$, $\delta_1^f < 1$ and $a_1 = a_0$, and where $R_1^d = \delta_1^d \hat{R}_1^d < 1$ and $R_1^f = \delta_1^f \hat{R}_1^f < 1$. Consumption is $c_2 = c_1 = g_0 + a_0 r_1 < 1$, whereas without a run consumption would be $c_2 = c_1 = g_0 + a_0 r_2 \geq 1$ for $r_2 \geq 1$.

In contrast, liquidity-based bank runs do not occur when private digital currency is used. When there is a high amount of early consumers, $\lambda > \hat{\lambda}(r_2)$, the price level P_1 increases at $t = 1$, such that the bank and firm do not default on early consumer withdrawals at $t = 1$. There is sufficient rollover lending to firms that late consumers receive a greater amount by withdrawing at $t = 2$ than $t = 1$.

Proposition 9 *Using a hard currency, there are inefficient insolvency-based banks runs when $r_2 < 1$ for all λ, and liquidity-based bank runs for $\lambda > \hat{\hat{\lambda}}(r_2)$ when $r_2 \geq 1$. Relative to using a private digital currency, there is excessive inefficient liquidation for $\lambda \in (\vec{\lambda}, \check{\lambda}(r_2))$ when $r_2 \geq 1$, and there is less continuing investment, output, consumption, and risk-sharing between early and late consumers for all macro states ω.*

Lower initial investment Because of the excessive liquidation caused by a moderately high λ and complete liquidation and runs caused by a very high λ, there is a lower amount of initial asset investment a_0 than otherwise optimal. However, the lower initial asset investment does not eliminate excessive liquidation and bank runs, unless expected returns are low enough and liquidation costs are high enough such that the initial asset amount is zero.

6.3 Discussion

Historically, gold has acted as the traditional hard currency. Under a gold standard, countries only issued money fully backed and redeemable at a fixed conversion rate to gold. In more recent decades, countries such as many emerging market economies with high fiat inflation often adopt a hard currency by directly using, or else pegging their currency to, a stable foreign fiat currency. One example is dollarization of the economy, in which only U.S. dollars are used as money and the domestic currency is no longer used. Dollarization can occur through government mandate, such as the case of Ecuador, or through the economy shifting to only dollar use if the government does not try or have the ability to ban it. Countries also try to peg their domestic currency, such as to the U.S. dollar, as in the case of Argentina that attempted to sustain a currency board that only issued domestic currency one-for-one in exchange with the dollar.

A private digital currency can prevent excessive fiat inflation, as with a hard currency, but has a more flexible value that can reduce the extreme bank fragility caused by a hard currency, support greater long-term investment, and give better macro liquidity and asset risk sharing among consumers.

Initial loan and deposit rates We do not analyze the optimal $\hat{R}_1^f$ and $\hat{R}_1^d$, as the results would be not be qualitatively changed by considering $\hat{R}_1^f \gtrless 1$ or $\hat{R}_1^d \gtrless 1$. $\hat{R}_1^f < 1$ cannot hold, as firms would only hold goods $g_0 = 1$ at $t = 0$ and would have a positive profit $c_2^f > 0$ for all states ω. Moreover, $\hat{R}_1^f < 1$ does not clear the market for loans at $t = 0$ since firms would demand infinite L_0^f borrowing of goods from the bank. $\hat{R}_1^f > 1$ would not change the bank run results, and $\hat{R}_1^f$ would affect the ex-interim efficiency results only in a qualitative way and as relative to R_1^d in the ratio $\frac{R_1^d}{R_1^f}$, as according to equation (49). For $\hat{R}_1^d > 1$, there would be an increase in risk-sharing among early and late consumers, as according to equation (49), when runs do not occur, but there would be an increase in the occurrence of insolvency and liquidity based bank runs. For $\hat{R}_1^d < 1$, the converse would hold. However, $\hat{R}_1^d < 1$ could not hold in equilibrium in a more generalized setting that allowed late consumers at $t = 1$ to make an initial bank deposit for a return of $\frac{R_1^d}{\hat{R}_1^d}$ at $t = 2$, since consumers would then only hold goods and no deposits at $t = 0$.

7 Concluding remarks

With the heightening interest and concern about the potential impact on the financial system and economy that may come from fintech, understanding the financial fragility and effect on banking that major financial technologies may bring is crucial.

This paper examines the potential impact of digital currency on economic investment and the stability of the banking system. Either CBDC or private digital currency permits but does not necessarily lead to the ex-ante disintermediation of the banking system, because banks create maturity and risk transformation that increases long-term investment and consumption apart from their role of providing for payments. Banks are partially buffered from macro liquidity and investment risk for deposits of private digital currency and can be fully buffered for fiat deposits. The primarily disintermediation threat takes the form of digital currency runs into private digital currency or CBDC, which can create fragility of the banking system. To prevent the disintermediation of banks by CBDC or runs into CBDC, central banks need to appropriately manage interest rates in a dynamic manner on fiat reserves and CBDC.

There is an important trade-off between the features of privately issued digital currency, such as bitcoin, and fiat money whether in traditional form or as CBDC. Central bank discretion permits fiat inflation that buffers the economy and banking system from macro risks but also can lead to excessive distortionary fiat inflation. Private digital currency precludes fiat inflation but also creates more rigidity in the banking system.

Appendix: Proofs

Section 2: Real economy

Proof for Proposition 1. Necessary first order conditions and sufficient second order conditions hold for the optimization (1) and require binding budget constraints, which imply consumption equations (2) and (3), and first order conditions equations (4), (5) and (6). For $g_1^* > 0$ and $\lambda < \check{\lambda}(r_2)$ with $r^k = 1$ and $\frac{u'(c_1^*)}{\beta u'(c_2^*)} = 1$ written as $c_2^* = \beta^\gamma c_1^*$, and for $a_1^* > 0$ and $\lambda > \hat{\lambda}(r_2)$ with $r^k = \frac{r_2}{r_1}$ and $\frac{u'(c_1^*)}{\beta u'(c_2^*)} = \frac{r_2}{r_1}$ written as $c_2^* = c_1^*(\beta \frac{r_2}{r_1})^\gamma$, substituting for c_2^* into equations (2) and (3) and solving gives, respectively,

$$
\begin{aligned}
g_1^* &= \tfrac{(1-\lambda)g_0^* - \lambda/(\beta r^k)^\gamma a_0^* r_2}{1-\lambda+\lambda/(\beta r^k)^\gamma} = \tfrac{(1-\lambda)g_0^* - (\lambda/\beta^\gamma)a_0^* r_2}{1-\lambda+\lambda/\beta^\gamma} && (51)\\
\check{\lambda}(r_2) &\equiv \tfrac{g_0}{g_0+a_0 r_2/\beta^\gamma} = \tfrac{g_0}{g_0+a_0 r_2/(\beta r^k)^\gamma} \in (0,1) \\
\alpha_1^* &= \tfrac{\lambda a_0^* r_2/(\beta r^k)^\gamma - (1-\lambda)g_0^*}{\lambda a_0^*/(\beta r^k)^\gamma + (1-\lambda) r_1} = \tfrac{\lambda a_0^* r_2/(\beta r_2/r_1)^\gamma - (1-\lambda)g_0^*}{\lambda \alpha_0^*/(\beta r_2/r_1)^\gamma + (1-\lambda) r_1} && (52)\\
\hat{\lambda}(r_2) &\equiv \tfrac{g_0}{g_0+a_0 r_2/(\beta r_2/r_1)^\gamma} = \tfrac{g_0}{g_0+a_0 r_2/(\beta r^k)^\gamma}.
\end{aligned}
$$

Section 3: CBDC

Proof for Lemma 1. The three budget constraints in the bank's optimization (13) can be consolidated into a single budget constraint by substituting for L_0^f and L_1^f as:

$$(1-\lambda')DR_2^d(\cdot) \leq (R_1^f - \lambda' R_1^d)DR_2^f - (R_1^f - R_1^m)R_2^f M_0^b - (R_2^f - R_2^m)M_1^b - (R_1^f R_2^f - R_1^m R_2^m)M^b \quad \forall\omega. \tag{53}$$

The first order condition with respect to M_t^b for $t = 0, 1$ is $R_t^f(\omega) = R_t^m(\omega)$ for $t = 1, 2$ and $\forall\omega$, which implies the resulting budget constraint simplifies as equation 15.

Proof for Lemma 2. To be completed.

Proof for Proposition 2. For the bank, consumer, and firm optimizations (13), (19) and (20), necessary first order conditions and sufficient second order conditions hold and market clearing for goods at $t \in \{0, 1, 2\}$ requires that all budget constraints bind. The market equilibrium exists and is unique up to an indeterminate price level at $t = 0$, P_0, with equilibrium prices P_t at $t \in \{1, 2\}$ given by equations (22) and

(23). With $\beta = 1$, since the central bank's objective function is equivalent to that for banks, the expected utility of consumers EU, the central bank optimally sets its rates on reserves.

Loans to firms made at $t = 1$ have a real return $r_2^{f^*} \equiv \frac{R_2^{f^*}}{\Pi_2}$. The firm's first order conditions with respect to $\{g_t, a_t\}_{t \in \{1,2\}}$ determine $a_t = a_t^*$ and $g_t = g_t^*$ for $t \in \{0, 1\}$, where for $\lambda < \check{\lambda}(r_2)$, $r_2^f = 1$; for $\lambda \in (\check{\lambda}(r_2), \hat{\lambda}(r_2))$, $r^f \in (1, \frac{r_2}{r_1})$); and for $\lambda \geq \hat{\lambda}(r_2)$, $r^f = \frac{r_2}{r_1}$. Thus, $q_1 = q_1^* = g_0^* + a_1^* r_1 - g_1^*$, and $q_2 = q_2^* = (a_0^* - a_1^*) r_2 + g_1^*$. From the consumption equations (17) and (18) and prices in equations (22) and (23), consumption for early and late consumers can be solved as

$$c_1 = \frac{\delta_1^d D R_1^d}{P_1} = \frac{q_1}{\lambda} \tag{54}$$

$$c_2 = \frac{\delta_2^d D R_2^d}{P_2} = \frac{q_2}{1-\lambda}, \tag{55}$$

which since $q_t = q_t^*$ gives $c_t = c_t^*$ for $t \in \{1, 2\}$.

Since the nominal and real rates for loans equal those rates for bank reserves, $R_t^f = R_t^m$ for $t \in \{1, 2\}$, the central bank chooses its policy rates to equate the MRS $r^c(R_1^d, R_1^f, r^k)$ determined by equation (42) with the optimal MRS $r^{c^*}(\beta)$ given by equation (10), which from these two equations can be solved as

$$\frac{R_1^{m^*}}{R_1^{d^*}} = \lambda' + (1 - \lambda')(\beta r^k)^\gamma / r^k.$$

The bank's consolidated budget constraint (15) allows for solving

$$\frac{R_2^{m^*}}{R_2^{d^*}} = \frac{(1 - \lambda') R_1^{d^*}}{(R_1^{m^*}/R_1^{d^*} - \lambda')} = r^k R_1^{d^*} / (\beta r^k)^\gamma$$

With this $\frac{R_1^{m^*}}{R_1^{d^*}}$, $r^c(\frac{R_1^{m^*}}{R_1^{d^*}}) = r^{k^*}$, $\check{\lambda}^f(\frac{R_1^{m^*}}{R_1^{d^*}}) = \check{\lambda}^*$, $\hat{\lambda}^f(\frac{R_1^{m^*}}{R_1^{d^*}}) = \hat{\lambda}^*$, $g_1(\frac{R_1^{m^*}}{R_1^{d^*}}) = g_1^*$, $a_1(\frac{R_1^{m^*}}{R_1^{d^*}}) = a_1^*$.

Proof for Corollary 1. Consumption for late consumers withdrawing i) early at $t = 1$ equals that of early consumers, $c_1^\ell = c_1$, and ii) at $t = 2$ equals c_2, for c_1 and c_2 determined in the proofs for proposition 2 if $\beta = 1$ and proposition 4 if $\beta < 1$.

Proof for Proposition 3. To be completed.

Proof for Proposition 4. From the central bank's optimization (21), the first order condition with respect to R_2^m implies that $\acute{R}_2^m > R_2^{f^*}$. The bank's first order conditions with respect to L_1^f and M_1^b require $R_2^f = R_2^m$, hence $R_2^f > R_2^{f^*}$ and $r_2^f > r_2^{f^*}$. If $\beta < 1$ is unexpected, then $a_0 = a_0^*$ and $g_0 = g_0^*$ are unchanged. The firm's first order conditions imply that $g_1 \leq g_1^*$ and $a_1 \geq a_1^*$, with $\acute{q}_1 > q_1^*$ and $\acute{q}_2 < q_2^*$. If $\beta < 1$ is expected, the firm's first order conditions imply that $a_0 < a_0^*$ and $g_0 > g_0^*$, which implies that $\acute{q}_1 > q_1^*$ and $\acute{q}_2 < q_2^*$. Hence, in either case, $\acute{c}_1 > c_1^*$, $\acute{c}_2 < c_2^*$, and $\acute{\Pi}_2 > \Pi_2^*$. The policy rates, and initial and ongoing investment, are efficient conditional on the central bank discount factor β^{cb}:

$$\begin{aligned}
R_t^j(\omega, \beta^{cb}) &= R_t^{j^*}(\omega, \beta^\ell) \text{ for } j \in \{m, c\},\ t \in \{1, 2\} \\
a_0^{cb} &= a_0^*(\beta^{cb}) \\
g_0^{cb} &= g_0^*(\beta^{cb}) \\
\check{\lambda}^{cb}(r_2, a_0^{cb}) &= \check{\lambda}^*(r_2, \beta^{cb}) \in \{\check{\lambda}^{cb}(r_2, a_0^*), \check{\lambda}^*(r_2, a_0^*)) \\
\hat{\lambda}^{cb}(r_2, a_0^{cb}) &= \hat{\lambda}^*(r_2, \beta^{cb}) \in (\hat{\lambda}^{cb}(r_2, a_0^*), \hat{\lambda}^*(r_2, a_0^*)) \\
g_1^{cb}(\omega, a_0^*) &= g_1^*(\omega, \beta^{cb}) \in (g_1^{cb}(\omega, a_0^*), g_1^*(\omega, a_0^*)) \\
a_1^{cb}(\omega, a_0^{cb}) &= a_1^*(\omega, \beta^{cb}) \in (a_1^{cb}(\omega, a_0^*), a_1^*(\omega, a_0^*)).
\end{aligned}$$

Section 4: Private digital currency

Proof for Lemma 3. The private issuer's binding budget constraints for $t \in \{0, 1, 2)$ at each date τ can be consolidated as a single budget constraint, solved for $c_{2,\tau}^y$ as

$$c_{2,\tau}^y \leq g_0^y(1 - P_{0,\tau}/P_{2,\tau}) - (M_{2,\tau}^y - M_{2,\tau-1}^y)/P_{2,\tau},$$

and substituted to write the issuer's objective function as

$$\textstyle\sum_{\tau=0}^{\infty} E[(1 - P_{0,\tau}/P_{2,\tau})g_{0,\tau}^y - (M_{2,\tau}^y - M_{2,\tau-1}^y)/P_{2,\tau}].$$

The first order conditions with respect to $g_{0,\tau}^y$ and $M_{2,\tau}$ are

$$\begin{aligned}
g_{0,\tau}^y &: E[\frac{1}{P_{2,\tau}}] = \frac{1}{P_{0,\tau}} \\
M_{2,\tau}^y &: E[\frac{1}{P_{2,\tau+1}}] = E[\frac{1}{P_{2,\tau}}].
\end{aligned}$$

Subsituting for prices into the issuer's binding budget and feasibility constraints, $M^y_{2,\tau} = M^y$, $M^y_{1,\tau} = M^y_{0,\tau} = 0$, $g^y_{t,\tau} = q^y_{t,\tau} = \frac{M^y}{P^y_{0,\tau}}$ for $t \in \{0, 2\}$, and $c^y_{2,\tau} = 0$ for all τ. There is no inflation for private digital currency across dates: $\Pi_{1,\tau}(\omega_\tau)\Pi_{2,\tau}(\omega_\tau) = 1$, $P_{2,\tau}(\omega_\tau) = P_{0,\tau}$, and $P_{0,\tau+1} = P_{2,\tau}(\omega_\tau) = P_{0,\tau}$ for all dates τ and states ω_τ.

Proof for Lemma 4. Using the single consolidated bank budget constraint in equation (53), the first-order condition and complementary slackness for the bank's optimization with respect to M^b_t for $t = 1, 2$ require $R^f_1 \geq R^m_1 = 1$ and $R^f_2 \geq R^m_2 = 1$, which bind if $M^b_t > 0$ for $t = 0$ and $t = 1$, respectively. Hence, $(R^f_1 - R^m_1)M^b_0 = 0$ and $(R^f_2 - R^m_2)M^b_1 = 0$, which substituting into equation (53) and simplifying gives (15). Note that if firms were considered for holding private digital currency M^f_t at $t \in \{0, 1\}$, the firm's first-order conditions with respect to L^f_t and M^b_t for $t \in \{0, 1\}$ would require analogous as for the bank that $R^f_1 \geq R^m_1 = 1$ and $R^f_2 \geq R^m_2 = 1$, which bind if $M^f_t > 0$ for $t = 0$ and $t = 1$, respectively.

Proof for Lemma 5. The proof follows the proof for no consumption-based runs for fiat deposits in lemma 1. An amount up to $\lambda' - \lambda$ late consumers may withdraw and buy goods at $t = 1$, where $\lambda' \leq \check{\lambda}(r_2 < 1)$. Firms store goods g_1 goods. Consumption $c^\ell_2 = c^\ell_1 = c_1$ is unchanged. A fraction $\lambda' > \check{\lambda}(r_2 < 1)$ of withdrawals at $t = 1$ is not an equilibrium. For $\lambda' > \check{\lambda}(r_2 < 1)$, depositors consumption for depositors withdrawing at $t = 1$ and $t = 2$ would equal that of early and late types in the state with $\lambda > \check{\lambda}(r_2 < 1)$, in which $R^d_2 > R^d_1$ and $c^\ell_2 > c^\ell_1$. No goods need to be stored at $t = 1$, $g_1 = 0$, since there are not too many late types needing to consume out of the low asset returns at $t = 2$.

Proof for Proposition 5. Binding budget constraints consolidated for the bank and firm at $t = 1$ and at $t = 2$, and then consolidated for $t = 1$ and $t = 2$ together, are respectively:

$$t{=}1: \quad \lambda\delta^d_1\hat{R}^d_1 = (g_0 + a_1 r_1 - g_1)P_1 + M^b_0 - M^b_1$$

$$t{=}2: \quad (1-\lambda)\delta^d_2\hat{R}^d_2 = [g_1 + (a_0 - a_1)r_2]P_2 + M^b_1$$

$$t{=}1{,}2: \quad \lambda\delta^d_1\hat{R}^d_1 + (1-\lambda)R^d_2 = (g_0 + a_1 r_1 - g_1)P_1 + [g_1 + (a_0 - a_1)r_2]P_2 + M^b_0$$

With all late consumers withdrawing $w^v = 1$ and holding private digital currency, the bank would default at $t = 1$, $\delta^d_1 < 1$, since the bank's $t = 0$ loans to firms are too illiquid to receive their par value repaid back at $t = 1$ to pay out all withdrawals at $t = 1$, and in particular does not have enough private reserves held from $t = 0$ to pay all of the late consumer demands for withdrawals in the form of digital currency, since $1 - \check{\lambda}(r_2) \geq \dfrac{a_0 r_2}{g_0 + a_0 r_2}$.

First, we show using a proof by contradiction that for $r_2 < 1$, the firm defaults at $t = 1$ with $\delta_1^f \hat{R}_1^f < 1$ or at $t = 2$ with $R_2^f < 1$. Suppose instead that for $r_2 < 1$, $\delta_1^f \hat{R}_1^f \geq 1$ and $\hat{R}_2^f \geq 1$. The firm's budget constraints for $t = 0, 1, 2$ in equation set (20) can be combined into the single budget constraint $(\delta_1^f q_0^v P_0 \hat{R}_1^f - q_1^v P_1)\hat{R}_2^f \leq q_2^v P_2$. With $\delta_1^f \hat{R}_1^f \geq 1$ and $\hat{R}_2^f \geq 1$; and with $P_0 = 1$, $P_1 \nleq 1$ and $P_2 \leq 1$ from lemma (3), the single budget constraint can be written as $q_0^v \leq q_1^v + q_2^v$. Substituting with the firm's feasibility constraints in equation set (20) and simplifying, the budget constraint is $a_0(r_2 - 1) - a_1(r_2 - r_1) > 0$, which is a contradiction for $r_2 < 1$ since $a_0 > 0$.

The requirement that the bank and firm maximize repayment on deposits and loans, respectively, in case of a default can be written in the form of complementary slackness conditions. Specifically, for every state $(\lambda_\tau, r_{2,\tau})$ at each date τ, the complementary slackness condition for the bank is $(1 - \delta_1^d)\phi_1^d = 0$ for $\phi_1^d \in \{L_1^f, M_1^b\}$ and for the firm is $(1 - \delta_1^f)\phi_1^f = 0$ for $\phi_1^f \in \{L_1^f, g_1, a_1 - a_0\}$.

For the bank, consumer, firm and issuer optimizations (13), (19), (20) and (45), necessary first order conditions and sufficient second order conditions hold and market clearing for goods at $t \in \{0, 1, 2\}$ requires that all budget constraints bind. The market equilibrium exists and is unique, with equilibrium prices P_t at $t \in \{1, 2\}$ given by equations (22) and (23) and consumption given by equations (54) and (55).

The bank's first order conditions give the bank's Euler equation, $E[u'(c_1)\frac{1}{P_1}] = E[u'(c_2)\frac{1}{P_2}R_2^f]$ with complementary slackness condition $(R_2^f - 1)M_1^b = 0$, The first order conditions for the bank's optimization and determine deposit and loan rates $R_1^{d^*} = R_1^{f^*} = 1$, $R_2^{d^*} = \bar{r}_2$, which are equivalent, and $R_2^{f^*}$, which is different, to the rates for the fiat case. The firm's first order condition with respect to a_0 gives the firm's Euler equation, $E[\frac{R_2^f}{\Pi_2}] = \bar{r}$, which for the real return $r_t^f = \frac{R_t^{f^*}}{\Pi_t}$ with the nominal rate functions determined by the bank and the equilibrium price functions, gives $r_2^{f^*} = r_2^{f^*}$ .At $t = 1$, for $\lambda < \check{\lambda}(r_2)$, $r_2^{f^*} = 1$ and firms optimally store $g_1 > 0$. For $\lambda \in [\check{\lambda}(r_2), \hat{\lambda}(r_2)]$, $r_2^{f^*} \in [1, \frac{r_2}{r_1}]$ and firms do not store or liquidate with $g_1 = a_1^* = 0$. For $\lambda > \hat{\lambda}(r_2)$, $r_2^{f^*} = \frac{r_2}{r_1}$ and firms partially liquidate $a_1^* > 0$. For all states ω, firms have zero consumption: $c_2^f = 0$.

Substituting with equilibrium prices into the budget constraints for the consumer, bank, firm, and issuer; applying market clearing conditions; and simplifying to solve for defaults; the equilibrium at $t = 1$ has no defaults with $\delta_1^d = \delta_1^f = 1$ for all λ, r_2, and at $t = 2$ has no absolute default with $\delta_2^d\, \hat{R}_2^d \geq 1$ if either $r_2 \geq 1$ or $\lambda(r_2) > \check{\lambda}(r_2)$ and has an absolute default with $\delta_2^d\, \hat{R}_2^d = g_0 + a_0 r_2 < 1$ if $\lambda \leq \check{\lambda}(r_2 < 1)$, where $\delta_2^d\, \hat{R}_2^d < 1$, since $g_0 + a_0 = 1$.

Proof for Corollary 2. To be completed.

Proof for Proposition 6. To be completed, following the proof for fiat deposits in proposition 2.

Section 5: Consumer lending

Proof for Proposition 7. If a consumer lends private digital currency at $t = 0$, the contracted return is $\hat{R}_1^f \geq 1$ since the loan market is competitive at $t = 0$, and can be rolled over at $t = 1$ for a return of R_2^f at $t = 2$. The firm uses the direct loan to buy $q_0 = L_0^f = 1$ goods to store and invest at $t = 0$, $g_0 + a_0 = q_0$, and sells its output of goods to repay its loan at either $t = 1$ and $t = 2$. An early consumer does not rollover any amount of the loan to the firm at $t = 1$. The firm's budget constraint at $t = 1$ is $\delta_1^e \hat{R}_1^f = q_1^e P_1$. The amount of goods the firm sells to repay its loan is $q_1^e = g_0 + a_1^e r_1 - g_1^e - c_2^f$. Even by fully liquidating its assets, with $a_1^e = a_0$, the maximum the firm can sell is $q_1^e = g_0 + a_0 r_1 \leq q_0 \leq 1$. The return that the firm repays is $\delta_1^e \hat{R}_1^f = q_1^e P_1 \leq 1$. Since $\hat{R}_1^f \geq 1$, $\delta_1^e < 1$ if either $q_1^e P_1 < 1$ or $\hat{R}_1^f > 1$. In particular, if the firm invests any amount $a_0 > 0$, then $q_1^e < 1$ and the firm defaults, $\delta_1^e < 1$, requiring full asset liquidation $a_1^e = a_0$. The early consumer has a real return in terms of consumption of $c_1 = \dfrac{\delta_1^e \hat{R}_1^f}{P_1} = q_1^e \leq 1$, with $c_1 < 1$ if $a_0 > 0$.

Proof for Corollary 3. To be completed.

Proof for Lemma 6. If the central bank has a short-term bias $\beta < 1$, consumers hold private digital currency, in the form of bank deposits, and do not hold fiat money deposits or public digital currency in any form. With public digital currency, inflation at $t = 2$ of any date τ, $\Pi_{2,\tau} = \frac{P_2}{P_1}$, is independent of D. Hence, the firm's real return $r_2^f \equiv \frac{R_2^f}{\Pi_2}$ is independent of D, which implies that for $t \in \{1, 2\}$, q_t, and thus c_t given by equations (54) and (55), are independent of D. With public digital currency, equilibrium prices at $t \in \{1, 2\}$ are

$$\begin{aligned} P_1 &= \tfrac{\lambda(DR_1^d + M_0^c R_1^c) + (1-\lambda)(M_0^c R_1^c - M_1^c)}{q_1} \\ P_2 &= \tfrac{(1-\lambda)(DR_2^d + M_1^c R_2^c)}{q_2}. \end{aligned}$$

Hence, $\frac{R_t^d}{P_t} > \frac{1}{P_t}$ for $t \in \{1, 2\}$, which implies from the consumer's first order conditions that $M_0^c = M_1^c = 0$.

Proof for Proposition 8. To be completed.

Section 6: Hard currency

Proof for Proposition 9. The equilibrium is determined for bank, firm, and consumer quantities and returns following the proof for lemma 5 with macro state ω but where prices $P_{t,\tau} \equiv 1$ for all $t \in \{0, 1, 2\}$ and all τ. The bank's first order conditions give the bank's Euler equation, $E[u'(c_1)] = E[u'(c_2) r_2^f\ ]$ with complementary slackness condition $(r_2^f - 1) m_1^b = 0$ and loan and deposit returns $r_1^f = 1$, r_2^f, R_1^d, and R_2^d. The firm's first order conditions give a_1, and g_1 and the firm's Euler equation, $E[r_2^f\ ] = \bar{r}$, which with r_2^f from the bank's first order condition, give a_0 and g_0. Consumers deposit $d = 1$ and have consumption $c_1 = \delta_1^d r_1^d$, $c_1^\ell = w\delta_1^d r_1^d$, and $c_2 = (1-w)\delta_2 r_2^d$. The budget constraints for the bank and firm can be consolidated as:

$$\begin{aligned} &t{=}0: && g_0 + a_0 + m_0^b = 1 \\ &t{=}1: && \lambda' \delta_1 r_1^d = g_0 + a_1 r_1 - g_1 + m_0^b - m_1^b \\ &t{=}2: && (1-\lambda')\delta_2 r_2^d = g_1 + (a_0 - a_1) r_2 + m_1^b \\ &t{=}1{,}2: && \lambda' \delta_1^d r_1^d + (1-\lambda')\delta_2 r_2^d = g_0 + a_0 r_2 - a_1(r_2 - r_1) + m_0^b \end{aligned}$$

First consider $w = 0$. The consolidated budget constraint for $t = 1$ requires that for the bank not to default, $\delta_1^d = 1$, then for $\lambda \leq \vec{\lambda} \equiv \frac{g_0}{R_1^d}$, there is an amount of combined goods and digital currency that is available to store until $t = 2$, $g_1 + m_1^b = \lambda r_1^d - (1 - \alpha^z) \geq 0$, whereas if $\lambda > \vec{\lambda} \equiv \frac{g_0}{R_1^d}$, there is a required amount of investment liquidation, $a_1 = \frac{\lambda r_1^d - (1-\alpha^z)}{r_1} = \frac{\lambda r_1^d - g_0 - m_0^b}{r_1} > 0$.

Substituting for these into the consolidated budget constraint at $t = 2$ and simplifying, the incentive constraint for the late consumer to not withdraw at $t = 1$, $R_2^d < r_1^d$, is violated if either $r_2 < \frac{r_1^d - g_0 - m_0^b}{a_0}$ for any λ or if $r_2 < \frac{(1-\lambda) r_1^d r_1}{g_0 + m_0^b + a_0 r_1 - \lambda r_1^d}$ for the following three cases. For $\lambda > \hat{\lambda}^z(r_2)$, the amount of liquidation a_1 is so large that the bank has a large enough default at $t = 2$ such that the late consumer's incentive constraint is violated, $\delta_2 c_2 < c_1$. Late consumers run the bank and fully withdraw at $t = 1$, $w = 1$. The bank and firm default and fully liquidate assets at $t = 1$, $\delta_1^d < 1$, $\delta_1^f < 1$, and $a_1 = a_0$. Based on $c_1 = 1$, for $\lambda \in (\hat{\lambda}^d, \hat{\lambda}(r_2))$, where $\hat{\lambda}^d \equiv g_0$ and $\hat{\lambda}^d \in (\check{\lambda}(r_2), \hat{\lambda}(r_2))$, there is excessive liquidation required for the firm and bank not to default at $t = 1$, which gives suboptimal consumption for late consumers: $a_1 = \dfrac{\lambda - \hat{\lambda}^d}{r_1} > a_1^* = 0$ and $c_2 < c_2^*$. For $\lambda \in [\check{\lambda}(r_2), \hat{\lambda}(r_2)], r_2^{f^*} \in [1, \frac{r_2}{r_1}]$, firms do not store or liquidate with $g_1 = a_1^* = 0$. For $\lambda > \hat{\lambda}(r_2)$, $r_2^{f^*} = \frac{r_2}{r_1}$ and firms partially liquidate $a_1^* > 0$. For all states ω, firms have zero consumption: $c_2^f = 0$.

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