Human Capital Investment and the Completion of Risky R&D Projects

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Abstract

We consider a firm that employs human capital to make a technological breakthrough. Since the probability of success of the breakthrough depends on the current stock of human capital the firm has an incentive to expand its human capital stock. The present value of the patent is stochastic but can be observed during the R&D phase of the project. The exogenous value of the patent determines the firm's decisions to invest in human capital, to abandon the project if necessary, and to invest in marketing the new product. We study the corresponding optimal stopping times, determine their value and risk consequences, and derive optimal investment in the stock of human capital. While optimal investment in human capital is very sensitive to its productivity do increase the probability of a breakthrough it is insensitive to changes in the volatility of the present value of the patent. The value of the firm is driven by fixed labor costs that occur until the breakthrough is made, the call option to invest in human capital and market the product, and the put option to abandon the project. These options together with labor costs' based operating leverage determine the risk dynamics. Risk varies non-monotonically with the stochastic value of the patent and is U-shaped.

1 Introduction

The analysis of investments in research and development (R&D) is a difficult task since R&D spending is exposed to multiple sources of uncertainties. If a pharmaceutical company invests in the development of a drug it not only faces the uncertainty about the technological breakthrough but also the uncertainty about the market conditions (expressed in terms of the present value of the patent) for selling the drug once the breakthrough is done. R&D investments are therefore guided by technological and market uncertainties.[1] These two types of uncertainties interact in a complex way making the firm's R&D investment decision a challenge to management.

Management is able to react to these sources of uncertainty by appropriate investment and exit strategies. The uncertainty of making the breakthrough depends among other things on the existing stock of human capital available in the research labs. Hence, management indirectly is able to control the success of the R&D investment. Prior to making the breakthrough management has the option to abandon the R&D project altogether and exit the market. Once the breakthrough is made the uncertainty about the stochastic value of the patent can be handled by either choosing not to market the product at all (abandon the patent) or by investing in the introduction of the product in the market place. These set of available choices to management demonstrates that R&D decisions are characterized by a complex interaction of investment and exit options.

The purpose of this paper is to analyze this sequence of real options and derive its implications for both the value of the company and corresponding risk dynamics within a simple analytical model. We assume that management's decisions are driven by an observable but stochastic patent value that follows a Geometric Brownian Motion and an exponential distribution for the completion date of the project. The hazard rate of this distribu-

[1] In case there is a patent race and several firms compete for the completion of a R&D project in addition firms face strategic risk that arises from the competitive interactions of the rival firms. While this is an important characteristic of R&D investment it will not be considered in this paper.

tion is assumed to depend on the existing stock of human capital present in the company. Hence the probability of making the breakthrough within the next small increment of time conditional on not having made the breakthrough up to today depends on the stock of human capital and therefore can directly be controlled by investment in HR. The decision to invest in human capital is triggered by a threshold level of the stochastic patent value that makes it attractive for the firm to increase its level of available skills. The exogenous patent value not only drives the investment decision to build up the stock of human capital it also determines the decision when to exit the market. This exit decision can be taken at different stages in the R&D process. The firm can either exit the market prior to having made the technological breakthrough or exit after the completion of the innovation but prior to marketing the product. Finally the stochastic value of the patent also drives the decision when to invest in marketing the product.

The complex decisions of R&D investments under technological and/or market uncertainties have been analyzed in numerous papers under the assumption of alternative market structures. Technological uncertainty and the economics of innovations are nicely summarized in the book by Kamien and Schwartz (1982). In most of these models the probability of making a breakthrough is either exogenous or depends on the current level of R&D investment. Fudenberg, Gilbert, Stiglitz, and Tirole (1983) are among the first to assume that the success probability depends on the stock of human capital available to a firm (see also the paper by Doraszelski (2003)). This implies that optimal R&D investment is the outcome of a dynamic trade-off between an increased completion rate and higher labor costs for the existing stock of human capital. In Jorgensen, Kort, and Dockner (2006) the exponential distribution of the completion date also depends on the stock of human capital. Using this assumption they study optimal financing structures and the role of venture capital in R&D investments.

While the first generation of innovation models concentrated on the modeling of technological uncertainty the next generation looked closely into the consequences of strategic competition among rival firms engaged in an R&D race. Reinganum (1982) studies an innovation race as a differential game in which competing firms invest in R&D in order to increase the probability of a breakthrough. She finds that competition for receiving a constant patent value substantially increases R&D investment and therefore the likelihood of success. Reinganum studies the patent race under the assumption of a constant patent value. Hence, she rules out market uncertainties. Patent races with a stochastic patent value are analyzed by Garlappi (2004) and Miltersen and Schwartz (2004). Garlappi (2004) studies the impact of competition on the risk premia of R&D ventures engaged in a multiple-stage patent race with technical and market uncertainty. He finds that a firm's risk premium decreases as a consequence of technical progress and increases when a rival pulls ahead. Miltersen and Schwartz (2004) analyze patent-protected R&D investment projects when there is (imperfect) competition in the development and marketing of the resulting product. They find that that R&D competition not only increases production and reduces prices, but also shortens the time of developing the product and increases the success probability. Weeds (2002) considers irreversible investment in competing research projects with uncertain returns and a winner-takes-all patent system. Firms face two uncertainties, probabilistic technological success and a stochastic patent value. In his framework the fear of preemption undermines the option value to delay investment so that two patterns of investment emerge, a preemptive leader follower and a symmetric equilibrium. In the preemptive equilibrium firms invest sequentially and option values are reduced by competition. In the symmetric equilibrium firms invest simultaneously and investment is delayed.

In a recent paper Miltersen and Schwartz (2007) study R&D investment with uncertain maturity and hence uncertain costs of completing the innovation. Technological uncertainty is modeled using an exponential distribution with a fixed intensity that can optimally be switched between two levels, high and low. When the firm chooses the high level of intensity fixed costs per unit of time are high, when it uses low intensity levels fixed costs are low. Additionally the firm has the option to abandon the project and leave the market when the patent value hits a low enough level or decide not to market the product after the breakthrough has been made.

We heavily build on the model of Miltersen and Schwartz (2007) and incorporate investment in human capital as the driving force for technological breakthrough. Specifically we assume that the breakthrough probability depends on the stock of human capital that is optimally determined by the firm exploiting a trade-off between increased costs of human capital and higher completion intensities. As in Miltersen and Schwartz (2007) the firm has the option to optimally abandon R&D efforts and shut down and not to market product once the breakthrough has been made. The distinguishing feature between our model and theirs rests on the optimal choice of the stock of human capital that influences the exponential success probability and its impact on the optimal exercise of the abandon and marketing option. Moreover, we focus on how the investment decision and the abandon and marketing options influence the firm's risk dynamics. This allows us to derive testable hypotheses about the relationship between human capital and the risk premia that can be earned in R&D intensive industries.

Endogenizing the investment decision in human capital has a profound impact on the firm value and risk dynamics of the company. We analytically show that the optimal firm value is a strictly convex increasing function of the patent value. It consists of the sum of the present value of future labor costs (operating leverage), the value of the investment to optimally expand the stock of human capital, and the value of the exit option. Firm risk is driven by these three value components. Fixed labor costs determine operating leverage that is risk increasing as is the option to choose optimal human capital levels and to market the product. The opportunity to exit the market if the present value of the patent turns out to be below an optimal threshold is risk reducing. During the period when the technological breakthrough has not been made operating leverage dominates firm risk when the patent value is low, and the investment option to choose an optimal level of human capital dominates when the patent value is high resulting in dynamic betas that are U-shaped. This suggests the testable hypothesis that risk premia for R&D intense firms are high when patent values are either very high or very low and are low for intermediate patent values. Moreover, we find that technological and market uncertainties have two very distinctive effects on the optimal level of human capital. While small changes in the intensity of the exponential success distribution that translate the existing level of human capital into a breakthrough probability have a huge impact on the optimal level of human capital, the volatility of the patent process has almost no effect on the optimal stock level. This implies that technological uncertainty is substantially more important for hiring skilled labor than market uncertainty. Hence regulatory actions that reduce the risk of future patent values does not seem to be as important than improving the productivity of skilled workers.

Our paper is organized as follows. In the next section we present the model and introduce two types of uncertainties, technological and market uncertainty. In Section 3 we derive firms values, optimal investment and exit triggers and dynamic betas. Section 4 is devoted to a numerical analysis in which we perform some comparative statics and Section 5 concludes the paper.

2 The Model

Consider a risk-neutral firm that carries out R&D activities for a project. If the project is completed successfully, the firm has the opportunity to market the product. A concrete example is a pharmaceutical company that allocates resources and expertise to develop a new drug or a vaccine. The drug can be used for treating a disease that is known to affect a certain portion of the population. If the company is successful in developing the drug, it faces the decision whether to market the drug. This is not a trivial decision since the demand for the drug may have declined by the time the company makes the breakthrough.[2] For ease of exposition, the subsections below describe the various aspects of the decision problem faced by the firm.

[2] A company that has developed a vaccine for the swine flu, for instance, might find that a rival has preempted or that the disease has faded away.

2.1 Investment in Human Capital

The firm uses its available human capital in its efforts to develop the product. Let k_0 denote the current level of human capital. To maintain its level of human capital, the firm incurs a variable cost of $w \geq 0$. This cost can be thought of as the wage rate paid to labor and expenses for periodic training activities.

The firm has the option to increase its level of expertise by investing in its human capital and thereby to increase the likelihood of a successful development of a product. The precise relation between the stock of human capital and the likelihood of a successful innovation will be developed in the next subsection. In general, depending on the expected value the new product will generate, the firm can invest or disinvest in its human capital several times during the course of the research activities. For illustrative purposes, however, we assume in this paper that the firm has the opportunity to increase its human capital stock to $k_1 > k_0$. The firm incurs an additional cost when it invests in human capital:

$$C(z) = cz^a \tag{1}$$

where $z \equiv k_1 - k_0$ denotes the addition to the current level of human capital and $a \geq 1$ is the curvature parameter. The cost function $C(z)$ can be thought of as summarizing the sum of costs for new technical equipment necessary for the hired labor, search costs, and any additional costs including training and orientation.

In practice, the firm's decision about its human capital has three dimensions. The first is the *timing* of the investment. At each point in time, the firm must decide whether to increase its level of expertise. The second dimension concerns the *level* of the investment. Conditional on the decision to invest, the firm must determine the optimal level of human capital. In this paper, we endogenize not only the timing of human capital investment but also the level of human capital after investment. A third aspect of human capital investment is when the investment becomes productive. One way to increase the level of expertise is to conduct training to familiarize the existing workforce with the latest developments in the field. This is often costly and can take time until the existing workforce becomes productive in the new techniques. Even in the case the firm hires new employees, it can still take time until the new workforce is oriented and familiarized with the research procedures of the company. Therefore, the firm must take the time lags involved in the investment process into account. Section 3.1 deals with this case. We now turn our attention to how the level of human capital influences the outcome of the R&D activities.

2.2 Innovation and Human Capital

Although the completion time of the R&D project is uncertain, the firm can affect it through its investment in human capital. To that end, define the random variable τ on the probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ as the time at which the innovation is made. Following a similar formulation in Jorgensen, Kort, and Dockner (2006), we assume that the hazard function is given by:

$$h^i(t) = \frac{f^i(t)}{1 - F^i(t)} \equiv \lambda k_i \tag{2}$$

where $f^i(t)$ and $F^i(t)$, $i \in \{0, 1\}$ are the density and the distribution functions of τ, respectively and $\lambda > 0$ is a fixed parameter. Equation (2) states that the probability of making the innovation in a short time interval dt is a function of the *level* of human capital stock the firm possesses. The parameter λ in this formulation can be interpreted as the effectiveness of human capital in employing the expertise within the firm. Specifically, it reflects factors such as the organizational structure of the firm, division of labor or whether the researchers are employed in their primary areas of expertise.

Since the hazard function uniquely determines the distribution of the random variable τ, $F^i(t)$ is given by the exponential distribution:

$$F^i(t) = 1 - e^{-\lambda k_i t} \tag{3}$$

Recall that the random completion time of the project is one of the two sources of uncertainty faced by the firm. The second source, pertaining to the market uncertainty, is discussed next.

2.3 The Patent and Product Marketing

If the firm successfully completes the project, it is entitled to a patent. If the firm decides to market the product, it receives a flow, $x(t)$, that corresponds to a rent equal to the present value of the patent. The value of the patent is assumed to follow a geometric Brownian motion:

$$dx(t) = \mu x(t)dt + \sigma x(t)dB(t) \tag{4}$$

where $dB(t)$ are the increments of a standard Brownian motion and μ and σ are the drift and volatility parameters. The drift parameter is assumed to be less than the risk-less rate, r.

Although equation (4) implies that the expected value of the patent increases over time, it does not necessarily follow that the firm immediately markets the new product as soon as the innovation is made. This is because marketing the product involves additional costs, which we denote by $I > 0$. For instance, a pharmaceutical firm in the United States, incurs expenses necessary to obtain the FDA approval as well as advertising and promotion expenditures. The firm, therefore, markets the product only if the value of the patent at the time of innovation, x_τ, exceeds the cost of marketing the product, I. Otherwise, the firm loses the opportunity to market the product.

To simplify the derivations in the subsequent sections, we scale the patent value and the cost of final investment. Let $y(t) \equiv \frac{x(t)}{I}$. By Itô's lemma, the scaled process $y(t)$ follows:

$$dy(t) = \mu y(t)dt + \sigma y(t)dB(t) \tag{5}$$

The firm's payoff after the innovation has been made can now be characterized as:

$$\pi(y) = max[y_\tau - 1, 0] \tag{6}$$

We have now outlined the basic structure of the model. What remains is the description of the firm's objective function and the decision problem. The next subsection deals with these issues.

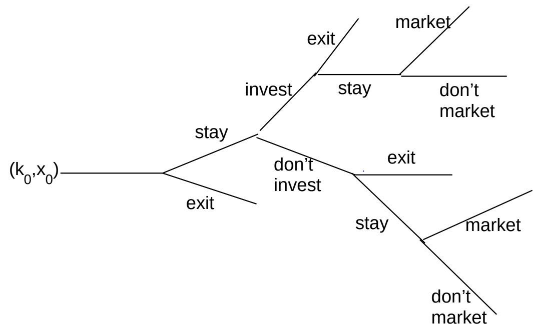

Figure 1: *Graphical Overview of the Decision Process.*

2.4 The Decision Problem

Before stating the decision problem of the firm, it is useful to describe the states of the model and how the firm moves from one state to another. Figure 1 presents a graphical overview of the model. At any time $t \geq 0$, the firm is either in the research stage[3] or in the marketing stage. Furthermore, in each stage, the firm can be in one of three states. The firm can operate with a human capital level of k_0 or k_1 or it can abandon the project altogether if the patent value is sufficiently low.[4]

[3]The research stage can also be thought of as the product development stage in other contexts. One example could be the development of a new beverage at Coca Cola.

[4]Although the value of the patent is also a state variable, it will be much easier to describe the model in terms of the states of level of human capital. The changes in the level of human capital, however, will be linked to the movements in the patent value.

The firm starts the project with a given level of human capital, $k_0 > 0$ and the patent value, y_0. At each point in time, the firm decides whether to continue the research or abandon it. The firm has the option to abandon the research phase independent of whether it has already invested in human capital. In addition, during the course of the research phase, the firm might find it optimal to increase the level of human capital to a level k_1. Assume, for the time being, that the new workforce becomes productive as soon as the investment is made. An increase in the value of the patent motivates the firm to invest in human capital and increases the likelihood of making the breakthrough. The increased likelihood of making the innovation, in turn, leads to a higher probability of immediately marketing the product once the breakthrough occurs. If investment in human capital occurs before the innovation, the firm moves to the second stage (i.e. the marketing stage) with a human capital level of k_1. If, on the other hand, the innovation occurs before it is worthwhile to undertake an investment in human capital, the firm ends up in the marketing stage with a human capital level of k_0. In the marketing stage, the firm simply decides whether to market the product. Although the firm can again be in either one of the states k_0 or k_1, the states are immaterial to the marketing decision and the decision is motivated solely by the value of the patent.

As discussed above, if the value of the patent increases sufficiently, the firm might find it worthwhile to invest in human capital and thereby increase the probability of making the innovation. We denote by τ_i, defined on the probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ and adapted to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$, the optimal time of investment in human capital

$$\tau_i = inf \{t \geq 0 | y_t \geq y_i\} \tag{7}$$

where y_i denotes the patent value that triggers investment.

The firm has the option to abandon the project any time during the research phase. Formally, let the adapted stopping time τ_e denote the time at which the firm abandons the research:

$$\tau_e = inf\left\{t \geq 0 | y_t \leq y_e\right\} \tag{8}$$

where y_e denotes the critical value that leads to the abandonment decision.

We are now in a position to state the objective function of the firm. The firm chooses the optimal time to invest in human capital as well as the optimal abandonment time to maximize its expected present value conditional on the information it holds:

$$V_0(y, k_0) = \max_{\tau_i, \tau_e, k_1} \mathbb{E}\left\{ \int_0^{\tau_m} -wk_0 e^{-rt} dt + \mathbb{I}_{\tau_i < \tau \wedge \tau_e} e^{-r\tau_i} \left[V_1(y, k_1) - C(z)\right] \right. \tag{9}$$
$$\left. + \mathbb{I}_{\tau < \tau_e \wedge \tau_i} e^{-r\tau} \pi(y) | \mathcal{F}_0 \right\}$$

where $V_0(y, k_0)$ and $V_1(y, k_1)$ are the value of the firm operating with the human capital levels of k_0 and k_1, respectively, $\mathbb{I}_{\tau_i<\tau}$ and $\mathbb{I}_{\tau<\tau_i}$ denote indicator functions that equal one when the respective conditions hold and $\tau_m \equiv min[\tau, \tau_i, \tau_e] \equiv \tau \wedge \tau_i \wedge \tau_e$ denotes the time at which the firm either changes state or moves to the marketing stage. Equation (9) states that the firm incurs an expense of wk_0 until τ_m is realized. If $\tau_i < \tau \wedge \tau_e$, the firm changes *state* by investing in human capital. This is captured by the second term in the expectation. If, on the other hand, the innovation is made while the firm is operating with k_0, the firm moves into the marketing *stage* in which it has to decide whether to market the product. This is reflected by the last term in the expectation. Finally, if the firm abandons the project, its value is assumed to be 0 and the firm does not have an re-entry option.

3 Optimal Human Capital Investment

In order to obtain a solution to the firm's problem in equation (9) subject to the state equation (5), we proceed as follows. We divide the problem into three parts. First, conditional on not having made the innovation and assuming that the firm has already undertaken the human capital investment, we determine the value of the firm for any level k_1. In this case, the firm has one strategic decision to make, namely when to optimally abandon the research stage. This corresponds to the value function $V_1(y, k_1)$ in equation (9). Once the value of the firm with the human capital level k_1 is determined, the next step is to derive the optimal level human capital. Finally, we return to the initial problem posed in equation (9) and characterize the optimal time of the investment in human capital as well as the optimal time to abandon the project.

The value of the firm after the investment in human capital has two components. The first is the cost per unit of time of maintaining the stock of human capital. The second component is the payoff at the breakthrough date, $\pi(y)$. The value of the firm can, therefore, be written as:

$$V_1(y, k_1) = \max_{\tau_e} \mathbb{E} \left\{ \int_{\tau_i}^{\tau \wedge \tau_e} -wk_1 e^{-rt} dt + \mathbb{I}_{\tau < \tau_e} e^{-r\tau} \pi(y) | \mathcal{F}_{\tau_i} \right\}. \tag{10}$$

Recall from the discussion in Subsection 2.3 that the value of the option to market the product depends on whether the value of the patent after the breakthrough is greater than the cost of marketing the product, normalized to 1. Therefore, we evaluate equation (10) in two regions based on whether $y(t) \leq 1$ or $y(t) > 1$. Proposition 1 states the first main result.

Proposition 1: Conditional on not having made the innovation, the value of the firm operating with the human capital level k_1 is given by:

$$V_1(y, k_1) = \begin{cases} \dfrac{-wk_1}{r + \lambda k_1} + A_1(k_1)y^{\alpha_1} + A_2(k_2)y^{\alpha_2}, & y_e \leq y < 1 \\ \dfrac{\lambda k_1 y}{r + \lambda k_1 - \mu} - \dfrac{(w + \lambda)k_1}{r + \lambda k_1} + B_2(k_1)y^{\alpha_2}, & 1 \leq y \end{cases} \tag{11}$$

where $A_1(k_1)$, $A_2(k_1)$ and $B_2(k_1)$ are constants given by:

$$\left.\begin{aligned} A_1(k_1) &= \frac{-\alpha_2 w k_1}{(r + \lambda k_1)(\alpha_1 - \alpha_2)} \frac{1}{y_e^{\alpha_1}}, \\ A_2(k_1) &= \frac{\alpha_1 w k_1}{(r + \lambda k_1)(\alpha_1 - \alpha_2)} \frac{1}{y_e^{\alpha_2}}, \\ B_2(k_1) &= A_1(k_1) + A_2(k_1) - \frac{\lambda k_1 \mu}{(r + \lambda k_1 - \mu)(r + \lambda k_1)}. \end{aligned}\right\} \tag{12}$$

$\alpha_1(k_1) > 1$ and $\alpha_2(k_1) < 0$ are the roots of the equation:

$$\frac{1}{2}\sigma^2\zeta(\zeta - 1) + \mu\zeta - (r + \lambda k_1) = 0 \tag{13}$$

and the abandonment threshold, $y_e(k_1)$ is given by:

$$y_e(k_1) = \left\{ \frac{-\alpha_2 w(r + \lambda k_1 - \mu)}{\lambda k_1(r + \lambda k_1 - \mu\alpha_2)} \right\}^{1/\alpha_1}. \tag{14}$$

Proof: *See Appendix A*

Implicit in Proposition 1 is the assumption that the trigger to abandon, y_e, is less than the cost of marketing the product after the innovation. However, this need not necessarily hold. If the abandonment trigger is greater than the cost of marketing the product, the firm immediately undertakes the investment and markets the product after the innovation is realized. This implies that the region $y \in [y_e, 1]$ is irrelevant to the analysis. This case is analyzed in Appendix B.

Proposition 1 determines the value of the firm after human capital investment for a generic level of human capital, k_1. The next step is to determine the optimal level of k_1. The firm chooses the level of human capital so as to maximize the value of the firm after investment, $V_1(y, k_1)$ net of the investment cost, $C(k_1)$. Proposition 2 gives the optimal level of human capital.

Proposition 2: For a given level of the patent value, y, the optimal level of human capital, k_1^*, is determined by:

$$B_0^{'}(k_1^*) + B_2^{'}(k_1^*)y^{\alpha_2} + B_2(k_1^*)y^{\alpha_2} lny\alpha_2^{'}(k_1^*) - ac(k_1^* - k_0)^{a-1} = 0 \tag{15}$$

where

$$B_0(k_1) = \frac{\lambda k_1 y}{r + \lambda k_1 - \mu} - \frac{(w + \lambda)k_1}{r + \lambda k_1} \tag{16}$$

Proof: *The proof follows from differentiating the value function in Proposition 1 and the investment cost function in equation (1) with respect to* k_1

Note that the optimal level of human capital is derived by differentiating the value function in the region $y \geq 1$ only. This is justified because investment in human capital with $y < 1$ implies that the firm would not market the product once the breakthrough has been made. The optimality conditions have a very intuitive interpretation. The optimal level of investment requires that the costs for an additional unit of human capital is equal to the marginal contribution of this unit to the expected net present value of the patent including the value of the option abandon the project altogether. Hence, human capital levels will be larger the higher the net present value of the patent.

We now turn to the initial problem posed in equation (9). The solution of this problem follows the same line of arguments as that of equation (10). The firm invests and markets the product immediately upon the realization of the innovation if $y_\tau > 1$. When $y_\tau \leq 1$, the project is discarded. As opposed to the analysis of equation (10), however, the firm chooses the optimal time to invest in human capital as well as the optimal abandonment time. Proposition 3 summarizes the solution to the whole problem.

Proposition 3: Conditional on not having made the innovation, the value of the firm when it operates with a human capital level k_0 is given by:

$$V_0(y, k_0) = \begin{cases} \dfrac{-wk_0}{r + \lambda k_0} + C_1(k_0)y^{\gamma_1} + C_2(k_0)y^{\gamma_2}, & y_a \leq y \leq 1 \\ \dfrac{\lambda k_0 y}{r + \lambda k_0 - \mu} - \dfrac{w + \lambda k_0}{r + \lambda k_0} + D_1(k_0)y^{\gamma_1} + D_2(k_0)y^{\gamma_2}, & 1 < y \leq y_i \end{cases} \tag{17}$$

where $\gamma_1 > 1$ and $\gamma_2 < 0$ are the roots of the equation:

$$\frac{1}{2}\sigma^2\zeta(\zeta - 1) + \mu\zeta - (r + \lambda k_0) = 0 \tag{18}$$

and the set of constants $\{C_1, C_2, D_1, D_2\}$ and the abandonment and investment triggers, y_a and y_i are determined from:

$$\left.\begin{aligned}
&C_1 y_a^{\gamma_1} + C_2 y_a^{\gamma_2} - \frac{wk_0}{r+\lambda k_0} = 0\\
&\gamma_1 C_1 y_a^{\gamma_1 - 1} + \gamma_2 C_2 y_a^{\gamma_2 - 1} = 0\\
&C_1 + C_2 - D_1 - D_2 - \frac{\lambda k_0 \mu}{(r+\lambda k_0 - \mu)(r+\lambda k_0)} = 0\\
&\gamma_1 (C_1 - D_1) + \gamma_2 (C_2 - D_2) - \frac{\lambda k_0}{r+\lambda k_0 - \mu} = 0\\
&\Omega y_i + B_2 y_i^{\alpha_2} - D_1 y_i^{\gamma_1} - D_2 y_i^{\gamma_2} - \Gamma - c(k_1^*(y_i) - k_0)^a = 0\\
&\alpha_2 B_2 y_i^{\alpha_2 - 1} - (\gamma_1 D_1 y_i^{\gamma_1 - 1} + \gamma_2 D_2 y_i^{\gamma_2 - 1}) + \Omega = 0
\end{aligned}\right\} \tag{19}$$

where the constants Ω and Γ are defined as:

$$\left.\begin{aligned}
\Omega &= \frac{\lambda k_1^*(y_i)}{r+\lambda k_1^*(y_i) - \mu} - \frac{\lambda k_0}{r+\lambda k_0 - \mu}\\
\Gamma &= \frac{wk_1^*(y_i) + \lambda k_1^*(y_i)}{r+\lambda k_1^*(y_i)} - \frac{wk_0 + \lambda k_0}{r+\lambda k_0}
\end{aligned}\right\}$$

Proof: *See Appendix C*

Note that when solving the system of nonlinear equations in (19), the firm takes into account the optimal level of human capital after investment, k_1^*. Furthermore, as in Proposition 1, the working assumption in Proposition 3 is $y_a < 1$, that is, the abandonment trigger is less than the cost of making the final investment and marketing the product. The case $y_a > 1$ is treated in Appendix D.

3.1 Productivity Time Lags

This subsection introduces the phenomenon of time lags in productivity when the firm invests in its human capital. Such time lags may arise in the context of human capital investment from the activities that the firm has to undertake either to train the existing workforce or to orient and train the new workforce.

Suppose that when the firm invests in human capital, the expertise level does not immediately jump from k_0 to k_1. The firm operates with the human capital stock k_0 until the new workforce becomes productive. To model the time lag feature, we adopt the approach presented in Bar-Ilan and Strange (1996). Assume that the orientation of the new workforce takes a fixed amount of time, $h > 0$. We keep track of the remaining time until the workforce becomes productive through the variable $\theta \equiv h - t$, $t \in [0, h]$. The distribution of the time of innovation, τ, in the region $[\tau_i, \tau_i + h]$ is given by:

$$F^0(t) = 1 - e^{-\lambda k_0 t} \tag{20}$$

Incorporation of time lags in productivity introduces an intermediate step between the time the firm makes the investment and the time the human capital stock effectively becomes k_1. Let $V_2(y)$ denote the value of the firm in this intermediate state. While the firm is in this state, it can still make the innovation and, if the patent value is sufficiently small, it can abandon the research phase. Therefore, the value of the firm in the intermediate state can be written as:

$$V_2(y, k_1) = \max_{\tau_e} \mathbb{E} \left\{ \int_{\tau_i}^{\tau \wedge \tau_i + h} -wk_1 e^{-rt} dt + \mathbb{I}_{\tau_i + h < \tau} e^{-r(\tau_i + h)} \left[V_1(y, k_1) - C(z) \right] \right. \tag{21}$$
$$\left. + \mathbb{I}_{\tau < \tau_i + h} e^{-r\tau} \pi(y) | \mathcal{F}_{\tau_i} \right\}$$

Equation (22) states that, although the investment in human capital becomes productive with a time lag, the firm incurs the cost of maintaining the stock of human capital, w, immediately after the investment in human capital is undertaken. If the human capital becomes productive before the innovation is made, the firm value becomes $V_1(y, k_1)$ net of the cost of investment in human capital, $C(z)$. This is captured by the second term in equation (22). The final term accounts for the realization of the innovation before the new workforce becomes productive.

The decision problem of the firm at $t = 0$ must also acknowledge the time lags in productivity. As in Section 2.4, the firm determines when to invest in human capital and when to optimally abandon the research phase. The value of the firm at $t = 0$ can therefore be written as:

$$V_0(y, k_0) = \max_{\tau_i, \tau_e} \mathbb{E} \left\{ \int_0^{\tau_m} -wk_0 e^{-rt} dt + \mathbb{I}_{\tau_i < \tau \wedge \tau_e} e^{-r\tau_i} V_2(y, k_1) + \mathbb{I}_{\tau < \tau_i \wedge \tau_e} e^{-r\tau} \pi(y) | \mathcal{F}_0 \right\} \tag{22}$$

4 Numerical Analysis

The purpose of this section is twofold. First, we explore what drives the firm's human capital investment as well as the abandonment decision in the research phase. The decisions to invest in human capital and abandon the project are tied to two sets of parameters. The first pertains to the firm-specific factors such as the productivity of labor, cost of investment in human capital and the expected project completion time. The second set is related to the market factors controlling the expected patent value and its volatility.

The second purpose of this section is to investigate the implications of risky R&D and market uncertainty on both a firm's value and its risk dynamics over time. We are specifically interested how the abandonment and

Table 1: *Parameter Values*

Patent Parameters	Value	Human Capital Parameters	Value	Cost Parameters	Value
μ	0.03	λ	0.3	c	2
σ	0.20	k_0	1	a	1
r	0.06			w	0.06

human capital investment options affect the risk dynamics of the firm.

The factors that influence the decisions to invest in human capital and to abandon the research phase can be grouped into three sets of parameters. The first set is the parameters related to the patent value. These include the expected rate of change in the patent value, μ, the volatility of the patent value, σ and the prevailing market interest rate, r. Dixit (1989) shows that under uncertainty, the firms are more reluctant to either invest or abandon relative to the certainty case. This hysteresis effect is due to the value of postponing the decision until the market conditions become more favorable. We analyze whether the hysteresis effect still prevails when there is both economic uncertainty and technological uncertainty.

The decisions to invest and abandon are also a function of the firm-specific human capital factors. These include the productivity of human capital, λ, the current level of human capital, k_0 and the level of human capital expertise after investment, k_1. In addition to analyzing the hysteresis effect, the formulation in equation (2) allows us to investigate the impact of expected time to completion on the investment and abandonment decisions.

The final set of parameters consists of firm-specific cost parameters. We will investigate how the sunk cost of investment in human capital affects the investment and abandonment decisions. The baseline set of parameters

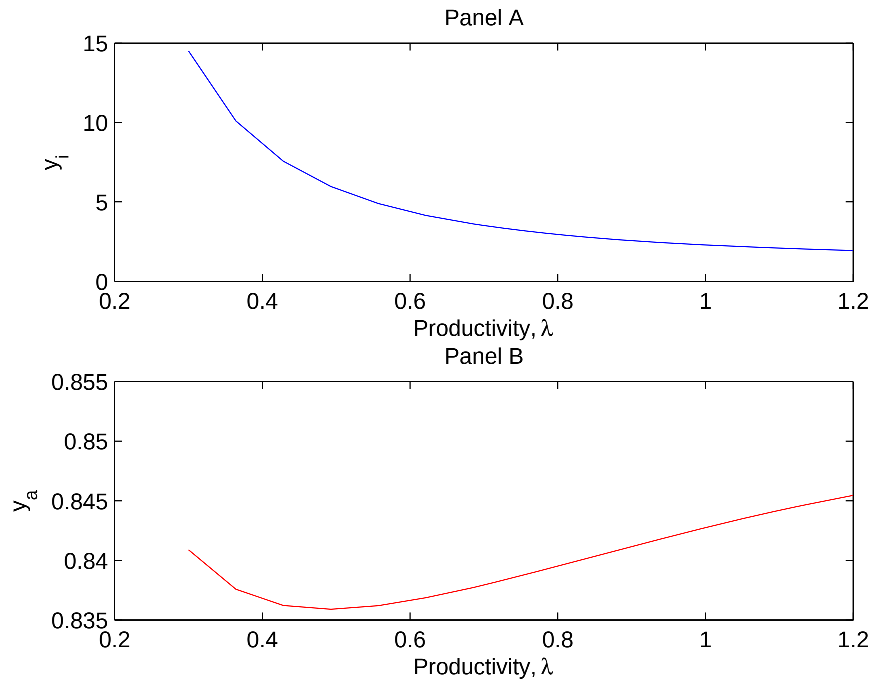

Figure 2: *Triggers and Human Capital Productivity*

are shown in Table 1. The comparative statics results are produced by numerically solving the nonlinear system of equations in Proposition 3.

4.1 Determinants of Investment and Abandonment

Figure 2 illustrates how the firm's investment and abandonment policies behave as a function of the human capital productivity, λ. Panel A of the figure shows that the firm invests sooner as the human capital productivity increases. This is because a higher human capital productivity increases the probability of making the innovation sooner and the firm can expect to obtain the marketing option at an earlier time. In addition, moving to the second stage (i.e. the marketing stage) sooner enables the firm to save the flow costs, $wk_i, i \in \{0, 1\}$.[5] The effect of the human capital productivity on

[5] The model assumes that there is no difference in productivity between existing human capital and acquired human capital. To the extent that the newly acquired expertise is at

the abandonment trigger, on the other hand, is nonmonotonic, as depicted in Panel B. For low values of λ, the abandonment trigger decreases. However, as the productivity parameter increases beyond 0.5, the firm abandons the research phase sooner. Furthermore, unreported comparative statics analysis shows that the abandonment trigger as a function of λ is sensitive to the specification of other parameters such as σ and μ.[6] For instance, when the volatility is increased to 0.4, the abandonment trigger is monotonically increasing and concave in the human capital productivity.[7]

Although the completion time of the research phase is stochastic, the firm can base its investment and abandonment policies on the expected time to completion. Given the distributional assumption in equation (2), the expected time to completion is $T_i = \frac{1}{\lambda k_i}$ when the firm operates with a human capital level $k_i, i \in \{0, 1\}$. There are two channels through which the expected time to completion can change. The first is the productivity of the human capital stock, λ. An increase in the productivity of human capital, *ceteris paribus*, shortens the expected time to completion of the research phase. We call this effect the *productivity channel*. The expected time to completion can also be shortened by investing in human resources and thereby increasing the expertise in the company. This is the *investment channel*. Figure 3 explores firm's investment decision as a result of a change in the productivity and the investment channels. The figure treats the level of human capital, k_1, also as an exogenous parameter. Panel A of the figure plots the investment trigger y_i against the average completion for varying levels of human capital productivity. The graph shows that higher average

least as productive as the existing human capital stock, this result is likely to be stronger.

[6]These graphs are available upon request from the authors.

[7]The behavior of the investment trigger, on the other hand, is robust to these alternative specifications.

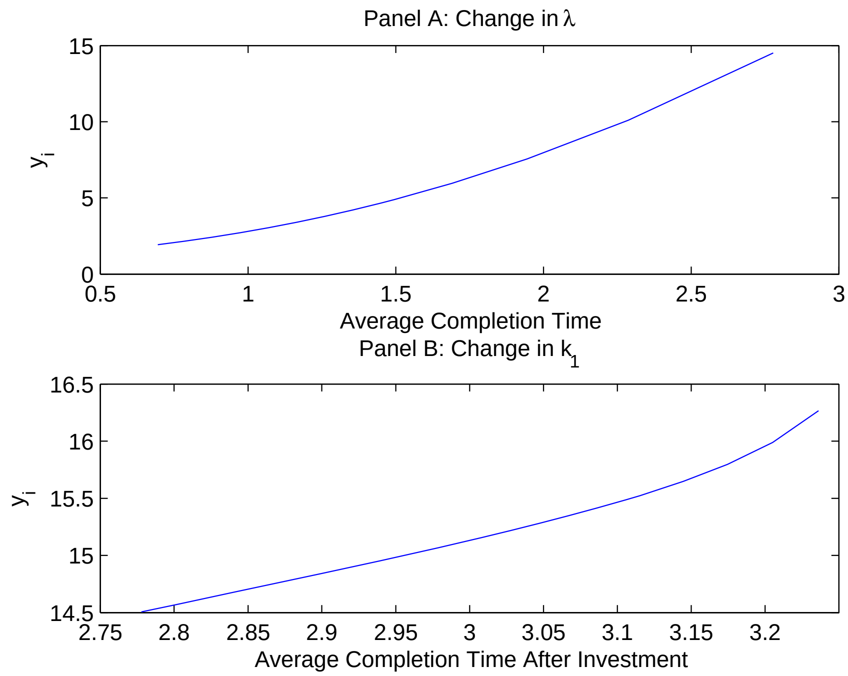

Figure 3: *Triggers and Expected Completion Time When Productivity Changes*

time to completion discourages the firm from undertaking the investment in human capital. Since Panel A holds the level of investment fixed, higher average completion time is equivalent to a low human capital productivity. In this case, the low productivity of human capital does not justify the costs of maintaining a high stock of human capital, leading to the reluctance in investment decisions. Panel B of Figure 3, on the other hand, holds the productivity constant and investigates the effect of different levels human capital investment. The firm is more willing to invest when investment substantially reduces the expected completion time after investment. This observation implies that the firm prefers lumpy investment to incremental investment.

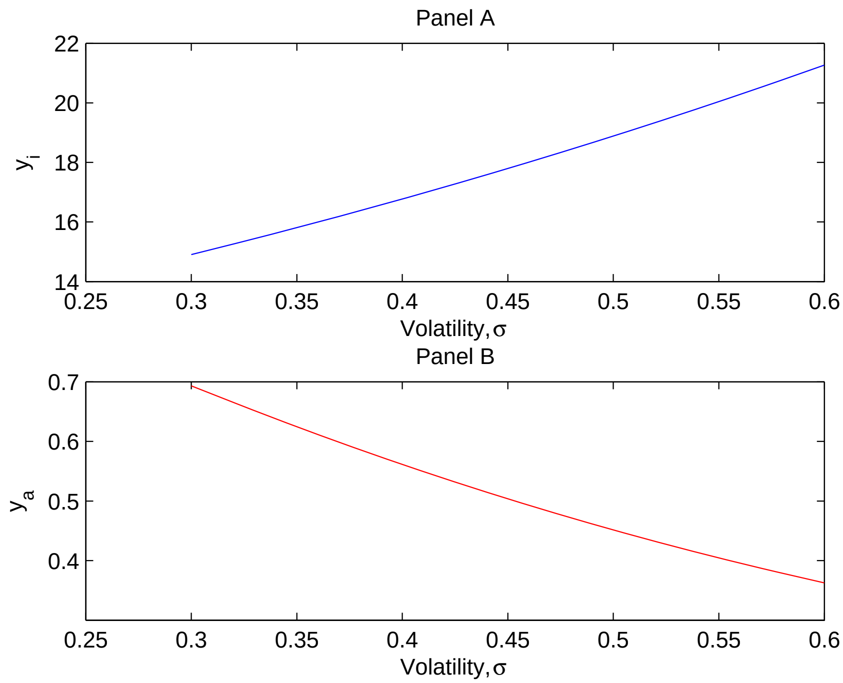

Figure 4: *Triggers and Volatility of the Patent*

Figure 4 shows investment and abandonment policies as a function of the patent volatility. In line with the standard real options models, a high volatility widens the difference between the investment and abandonment triggers. This is in line with the Dixit (1989) intuition and contrasts with the conclusion of Weeds (1999), who argues that the incorporation of the technological uncertainty as well as the economic uncertainty narrows the gap between the triggers. In our model, however, the decision to abandon is completely irreversible as opposed to her model, which considers costly switching options. Therefore, the firm in our model finds it optimal to postpone the irreversible abandonment decision when the volatility is high.

Finally, Figure 5 investigates the effect of the drift parameter on the investment and abandonment decisions. As Panels A and B show, the firm

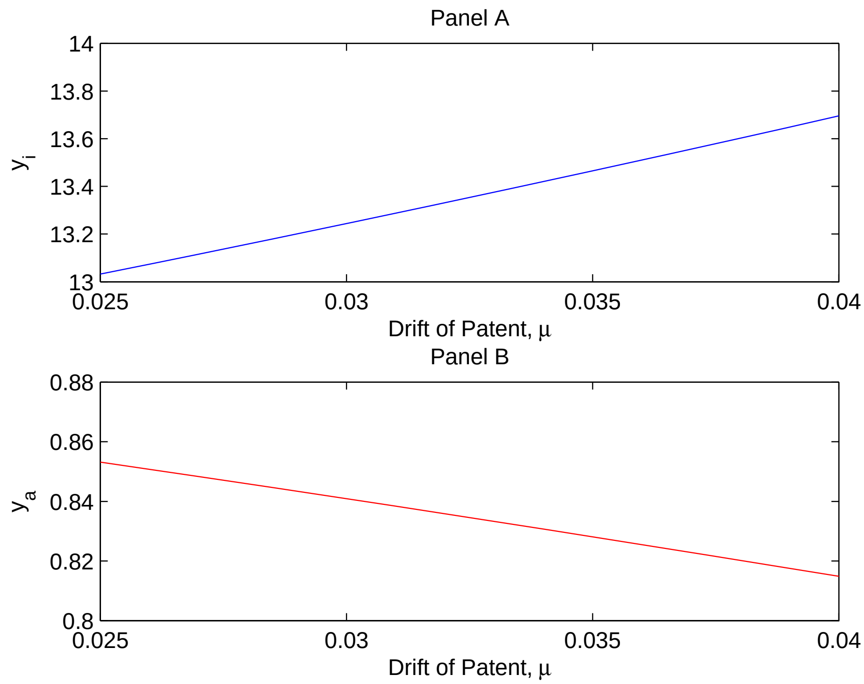

Figure 5: *Triggers and Drift of Patent*

becomes more reluctant to invest or abandon the research phase as the drift parameter, μ, increases. This result is intuitive. A higher drift parameter leads to a higher value of both investment and abandonment options. If the firm takes either decision too soon, it loses these option values. Therefore, a higher drift parameter leads to a postponement of the investment and abandonment decisions.

4.2 Determinants of Optimal Level of Human Capital

Table 2 and Table 3 explore the impact of uncertainty and sunk cost on the optimal level of human capital after investment. Table 2 also shows the corresponding abandonment and investment triggers. Recall that the firm faces two sources of uncertainty. The first source pertains to the uncertainty regarding the innovation time while the second is related to the volatility

Table 2: *Uncertainty and Optimal Human Capital Level*

$\lambda(\sigma = 0.2)$	k_1^*	y_e	y_i	$\sigma(\lambda = 0.3)$	k_1^*	y_e	y_i
0.1	2.3714	1.0768	91.3916	0.2	1.4483	0.9043	13.2440
0.3	1.4483	0.9043	13.2440	0.4	1.4482	0.6547	16.7722
0.6	1.0457	0.8437	4.5959	0.6	1.4469	0.4551	21.2785
0.7	1.0001	0.8381	3.7205	0.8	1.4413	0.3162	26.9707

of patent value. An inspection of Table 2 suggests that the two sources have strikingly different effects on the optimal level of human capital and investment timing. The table shows that for larger values of λ, the firm invests marginally but at a lower investment trigger whereas for smaller values of λ, the firm significantly increases its human capital, albeit at a higher investment trigger. This result can be understood through two angles. First, a higher λ implies a relatively more productive human capital already in place. This reduces the incentive to significantly increase the level of human capital. In other words, the firm relies more on the productivity channel than on the investment channel in order to make the innovation. Second, when λ is high, the expected time to completion is shorter. The firm is therefore reluctant to undertake a substantial costly expansion of the human capital. It is important to note that this result is partly driven by the assumption that the human capital will be irrelevant in the second stage in which the firm takes the marketing decision.

As opposed to the productivity parameter, λ, the patent volatility, σ, mainly affects the investment timing rather than the optimal level of human capital. Optimal human capital, k_1^*, remains relatively flat with respect to changes in σ while the gap between the investment and abandonment triggers widens as σ increases. As noted in the previous section, this is in line with the hysteresis effect where higher uncertainty leads to a larger

Table 3: *Sunk Cost and Optimal Human Capital Level*

	$a = 2$	$a = 1$		$a = 2$	$a = 1$
$\lambda(\sigma = 0.2)$	k_1^*	k_1^*	$\sigma(\lambda = 0.3)$	k_1^*	k_1^*
0.1	1.8064	2.3714	0.2	1.4800	1.4483
0.3	1.4800	1.4483	0.4	1.4800	1.4482
0.6	1.3197	1.0457	0.6	1.4795	1.4469
0.7	1.2996	1.0001	0.8	1.4774	1.4413

range of inertia. Furthermore, as Table 3 illustrates, these results are robust to different assumptions on the cost of human capital investment. Table 3 revisits the effect of λ and σ with varying curvature parameters. Although the effect of uncertainty remains the same, the range of optimal human capital is much narrower when a is higher since investment cost can become prohibitively high for large values of human capital investment when a is large.

4.3 Value and Risk Dynamics

In Figure 6, we plot the value of the firm as a function of the patent value. Each panel of the figure illustrates how the firm value is influenced by several parameters of interest. The firm is worthless below the respective abandonment triggers in all panels. The value is strictly increasing in the patent value for all $y > y_a$. The results in Panels B and C are consistent with the real options literature. The firm value increases in the volatility of the patent value, σ, as well as in the expected growth rate, μ. This is because a rise in both parameters increases the value of the options to postpone investment and abandonment decisions. The firm value is also increasing in the productivity of human capital, as shown in Panel A. The value added from higher productivity is particularly pronounced for high levels of the patent value. Since productivity is inversely related to the expected time to

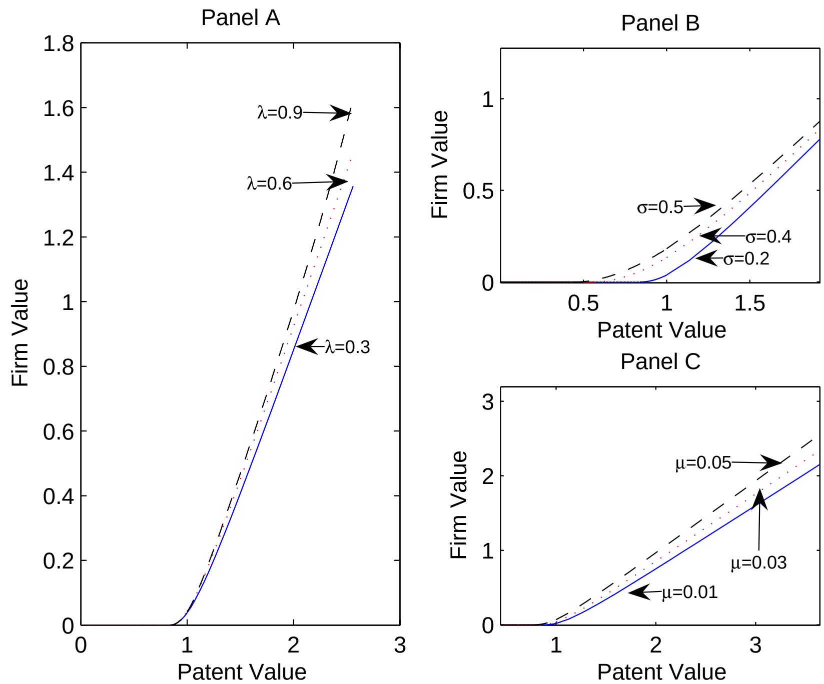

Figure 6: *The Patent and the Firm Value*

completion, Panel A also implies that the firm value increases as the expected time to completion is shortened through productivity improvements.

We now turn to the risk implications of the model. We define the firm risk as the sensitivity of the firm value to a given percentage change in the value of the patent. Proposition 4 specifies the firm risk explicitly.

Proposition 4: Let $\beta_i, i \in \{0,1\}$ denote the firm risk defined as:

$$\beta_i = \frac{\partial V_i(y,k_i)}{\partial y}\frac{y}{V_i(y,k_i)} \tag{23}$$

Then β_i are given by:

$$\beta_0(y,k_0) = \begin{cases} 1 + \dfrac{1}{V_0(y,k_0)}\left[\dfrac{wk_0}{r+\lambda k_0} + (\gamma_1 - 1)C_1 y^{\gamma_1}\right. & \\ \left. -(1-\gamma_2)C_2 y^{\gamma_2}\right], & y_a \le y \le 1 \\ 1 + \dfrac{1}{V_0(y,k_0)}\left[\dfrac{(w+\lambda)k_0}{r+\lambda k_0} + (\gamma_1 - 1)D_1 y^{\gamma_1}\right. & \\ \left. -(1-\gamma_2)D_2 y^{\gamma_2}\right], & 1 < y \le y_i \end{cases} \tag{24}$$

$$\beta_1(y) = \begin{cases} 1 + \dfrac{1}{V_1(y,k_1)}\left[\dfrac{wk_1}{r+\lambda k_1} + (\alpha_1 - 1)A_1 y^{\alpha_1}\right. & \\ \left. -(1-\alpha_2)A_2 y^{\alpha_2}\right], & y_e \le y \le 1 \\ 1 + \dfrac{1}{V_1(y,k_1)}\left[\dfrac{(w+\lambda)k_1}{r+\lambda k_1} - (1-\alpha_2)B_2 y^{\alpha_2}\right], & 1 < y \end{cases} \tag{25}$$

Proof: *The result is obtained by differentiating the value functions in Propositions 1 and 2 with respect to y.*

Proposition 4 shows that a firm's dynamic risk has several sources. The first component is the risk that emanates from the cash flows. This is normalized to 1 in our setting. The second component is the operating leverage and is captured by the first term in the square brackets in equations (24) and (25). Note that the operating leverage is associated with higher firm risk. The last two terms capture the risk that comes from the investment and abandonment options. Note that the investment and abandonment options have opposite implications for the firm risk. The investment option increases firm risk. The abandonment option, on the other hand, reduces the overall firm risk.

Figure 7 depicts the evolution of the firm risk. Panel A of the figure focuses on the risk dynamics before the investment in human capital has been

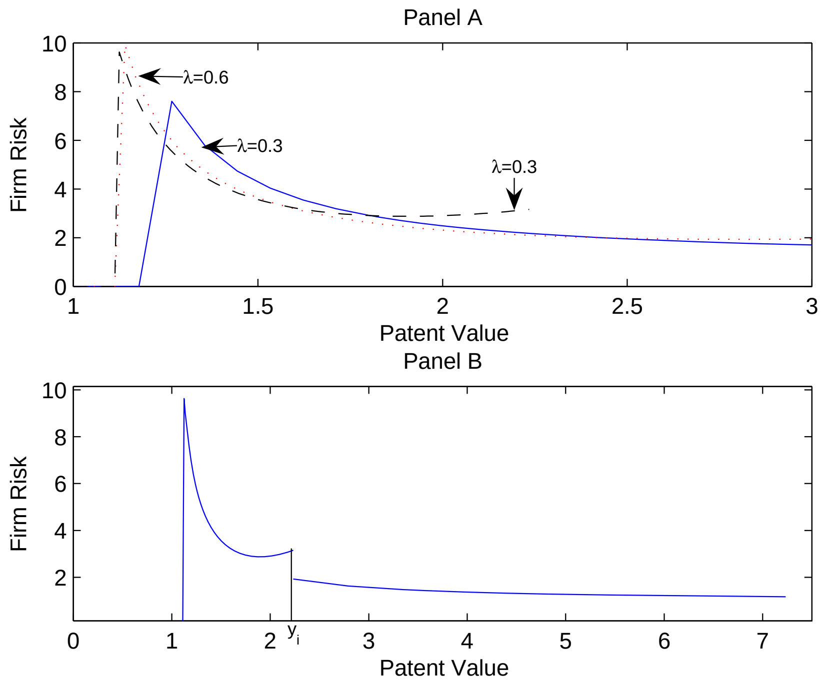

Figure 7: *The Patent and the Firm Value*

undertaken for various values of the productivity parameter λ. Before the investment in human capital, firm risk is driven by the options to invest in human capital and abandon the research as well as the operating leverage emanating from the existing level of human capital. As the patent value increases, the abandonment option has less value and the operating leverage becomes less significant. Hence, as the firm approaches the investment trigger, the investment option drives the firm risk. This leads the firm risk to increase. Note that firm risk is not monotonically related to the productivity parameter, λ. For small values of the patent value, firm risk decreases in λ. However, as the patent value approaches the investment trigger, firm risk is positively related to λ. This is because firm risk begins to increase sooner when λ is higher, which, in turn, is due to the relation between the investment trigger and *lambda*. Recall from the previous sections that a higher λ draws the investment trigger nearer.

On the other hand, when the patent value decreases, the investment option becomes less valuable and firm risk is driven by both the operating leverage and the abandonment option. Panel A of Figure 7 shows that the operating leverage dominates risk for moderate values of the patent value. This leads to an increase in firm risk. As the firm approaches the abandonment trigger, however, the option to exit dominates the operating leverage. This is reflected in the sharp decrease in firm risk near the exit trigger.

Panel B of Figure 7 illustrates the evolution of firm risk both before and after the investment in human capital. The panel reproduces the risk function from Panel A with $\lambda = 1$. Note that firm risk jumps at the moment of investment and significantly decreases. After the investment in human capital, firm risk monotonically decreases in the patent value. By investing in human capital, the firm foregoes the option to invest. This causes firm risk to be driven solely by the option to abandon and the operating leverage. As before, both sources of risk are decreasing in the patent value, leading to the monotonic relation in Panel B.

5 Conclusion

This paper explores the role of human capital in R&D investment under both technological and market uncertainty. The novel feature of our model stems from the assumption that the probability of success for the completion of an innovation depends on the stock of human capital and not the current level of investment. In this framework we derive optimal investment in human capital under the assumption of a stochastic patent value and study its implications for the value of the firm and its risk dynamics. Not surprisingly we find that the company value is a convex increasing function of the value of the patent. This result is entirely driven by the investment call option. Risk dynamics result in a U-shaped beta of the company. For small values of the patent, the firm faces high operating leverage arising from the fixed labor costs that cannot be substantially be reduced by the put option to exit the market. For large values of the patent risk is driven by option risk to exercise the investment option both to increase human capital and to market the product once the breakthrough has been made.

Appendix

Appendix A

Proof of Proposition 1: Let $V_{1b}(y, k_1)$ and $V_{1a}(y, k_1)$ denote the value functions in the regions $y \in [y_e, 1]$ and $y > 1$, respectively. Using Itô's lemma in the continuation region,[8] one can show that the value function of the firm satisfies the following system of ODE's:

$$\left.\begin{array}{ll} \frac{1}{2}\sigma^2 y^2 V_{1b}'' + \mu y V_{1b}' - (r + \lambda k_1)V_{1b} - w = 0, & y_e \leq y \leq 1 \\ \frac{1}{2}\sigma^2 y^2 V_{1a}'' + \mu y V_{1a}' - (r + \lambda k_1)V_{1a} + \lambda k_1(y-1) - w = 0, & 1 < y \end{array}\right\} \quad (26)$$

The two equations in (26) differ only in terms of their nonhomogeneous parts. The first equation reflects the fact that before the patent value reaches the cost of undertaking the final investment, 1, the firm simply incurs the cost of maintaining the stock of human capital. The second equation, on the other hand, states that, with an intensity λk_1, the firm is entitled to the payoff from marketing the product if the patent value is sufficiently high to justify the marketing.

[8]The continuation region is defined as the region in which the the firm remains in a given state. In the current discussion, the continuation region entails the value of the firm in state k_1 that has not yet made the innovation.

A particular solution for the system is given by the pair:

$$\left.\begin{array}{ll} V_{1b}(y,k_1) = -\dfrac{wk_1}{r+\lambda k_1}, & y_e \leq y \leq 1 \\ V_{1a}(y) = \dfrac{\lambda k_1 y}{r+\lambda k_1 - \mu} - \dfrac{(w+\lambda)k_1}{r+\lambda k_1}, & 1 > 1 \end{array}\right\} \tag{27}$$

Using equation (27), the general solution for the system (26) can be written as:

$$\left.\begin{array}{ll} V_{1b}(y,k_1) = -\dfrac{wk_1}{r+\lambda k_1} + A_1 y^{\alpha_1} + A_2 x^{\alpha_2}, & y_e \leq y \leq 1 \\ V_{1a}(y,k_1) = \dfrac{\lambda k_1 y}{r+\lambda k_1 - \mu} - \dfrac{(w+\lambda)k_1}{r+\lambda k_1} + B_1 y^{\alpha_1} + B_2 y^{\alpha_2}, & 1 < y \end{array}\right\} \tag{28}$$

where the set $\{A_1, A_2, B_1, B_2\}$ is the set of constants to be determined and $\alpha_1 > 1$ and $\alpha_2 < 0$ are the roots of the equation:

$$\frac{1}{2}\sigma^2\zeta(\zeta-1) + \mu\zeta - (r+\lambda k_1) = 0 \tag{29}$$

To obtain the set of constants as well as the optimal abandonment trigger, y_e, impose the following conditions:

$$\left.\begin{array}{l} V_{1b}(y_e,k_1) = 0 \\ V_{1b}^{'}(y_e,k_1) = 0 \\ V_{1b}(1,k_1) = V_{1a}(1,k_1) \\ V_{1b}^{'}(1,k_1) = V_{1a}^{'}(1,k_1) \\ \lim_{y\to\infty} V_{1a}(y,k_1) < \infty \end{array}\right\} \tag{30}$$

The last boundary condition in (30) implies that $B_1 = 0$. Plugging in the general solutions in (28) yields Proposition 1.

Appendix B

Suppose that the analysis of Proposition 1 yields $y_e > 1$. Then the firm always markets the product once the innovation has been made. Since the region $y \in [y_e, 1]$ is irrelevant to the analysis, we are left with:

$$\frac{1}{2}\sigma^2 y^2 \hat{V}_1^{''} + \mu y \hat{V}_1^{'} - (r+\lambda k_1)\hat{V}_1 + \lambda k_1(y-1) - w = 0, \quad y_e \leq y \tag{31}$$

To solve equation (31), impose the following boundary conditions:

$$\left.\begin{array}{l} \hat{V}_1(\hat{y}_e, k_1) = 0 \\ \hat{V}_1'(\hat{y}_e, k_1) = 0 \\ \lim_{y\to\infty} \hat{V}_1(y, k_1) = \dfrac{\lambda k_1 y}{r + \lambda k_1 - \mu} - \dfrac{(w+\lambda)k_1}{r + \lambda k_1} \end{array}\right\} \tag{32}$$

Solving equation (31) subject to (32) yields:

$$\left.\begin{array}{l} \hat{V}_1(y, k_1) = \dfrac{\lambda k_1 y}{r + \lambda k_1 - \mu} - \dfrac{(w+\lambda)k_1}{r + \lambda k_1} + \left[\dfrac{(w+\lambda)k_1}{r + \lambda k_1} - \dfrac{\lambda k_1 \hat{y}_e}{r + \lambda k_1 - \mu}\right]\left(\dfrac{y}{\hat{y}_e}\right)^{\alpha_2} \\ \hat{y}_e(k_1) = \dfrac{-\alpha_2((w+\lambda)k_1)(r + \lambda k_1 - \mu)}{\lambda k_1(1-\alpha_2)(r+\lambda k_1)} \end{array}\right\} \tag{33}$$

Appendix C

Proof of Proposition 3: To prove Proposition 3, we follow the same steps as in the proof of Proposition 1. In particular, in the continuation region, the value function satisfies:

$$\left.\begin{array}{ll} \dfrac{1}{2}\sigma^2 y^2 V_{0b}'' + \mu y V_{0b}' - (r+\lambda k_0)V_{0b} - w = 0, & y_a \le y \le 1 \\ \dfrac{1}{2}\sigma^2 y^2 V_{0a}'' + \mu y V_{0a}' - (r+\lambda k_0)V_{0a} + \lambda k_0(y-1) - w = 0, & 1 < y \end{array}\right\} \tag{34}$$

To solve the system (34), impose the following conditions:

$$\left.\begin{array}{l} V_{0b}(y_a, k_0) = 0 \\ V_{0b}'(y_a, k_0) = 0 \\ V_{0b}(1, k_0) = V_{0a}(1, k_0) \\ V_{0b}'(1, k_0) = V_{0a}'(1, k_0) \\ V_{0a}(y_i, k_0) = V_{1a}(y_i, k_1^*(y_i)) - c(k_1^*(y_i) - k_0)^a \\ V_{0a}'(y_i, k_0) = V_{1a}'(y_i, k_1^*(y_i)) \end{array}\right\} \tag{35}$$

The general solution for the system in (34) is given by:

$$\left.\begin{array}{l} V_{0b}(y, k_0) = \dfrac{-wk_0}{r+\lambda k_0} + C_1 y^{\gamma_1} + C_2 y^{\gamma_2} \\ V_{0a}(y, k_0) = \dfrac{\lambda k_0 y}{r + \lambda k_0 - \mu} - \dfrac{(w+\lambda)k_0}{r+\lambda k_0} + D_1 y^{\gamma_1} + D_2 y^{\gamma_2} \end{array}\right\} \tag{36}$$

Substituting equation (36) in the boundary conditions in (35) yields Proposition 3.

Appendix D

Suppose that the analysis in Proposition 3 yields $y_a > 1$. As discussed in Appendix B, this implies that the region $y \in [y_a, 1]$ is irrelevant to the analysis. The differential equation now reduces to:

$$\frac{1}{2}\sigma^2 y^2 \hat{V}_0'' + \mu y \hat{V}_0' - (r + \lambda k_0)\hat{V}_0 + \lambda k_0(y-1) - w = 0, \quad y_a \leq y \leq \hat{y}_i \tag{37}$$

To solve equation (31), impose the following boundary conditions:

$$\left.\begin{array}{l} \hat{V}_0(\hat{y}_a, k_0) = 0 \\ \hat{V}_0'(\hat{y}_a, k_0) = 0 \\ \hat{V}_0(\hat{y}_i, k_0) = \hat{V}_1'(\hat{y}_i, k_1^*(y_i)) - c(k_1^* - k_0)^a \\ \hat{V}_0'(\hat{y}_i, k_0) = \hat{V}_1'(\hat{y}_i, k_1^*(y_i)) \end{array}\right\} \tag{38}$$

The general solution to system (37) subject to (38) yields:

$$\hat{V}_0(y, k_0) = \frac{\lambda k_0 y}{r + \lambda k_0 - \mu} - \frac{(w + \lambda)k_0}{r + \lambda k_0} + E_1 y^{\gamma_1} + E_2 y^{\gamma_2} \tag{39}$$

The general solution in equation (39) subject to the conditions in (38) must be solved numerically.

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