Shareholder Monitoring and Risk Sharing

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Abstract

The owner of a firm can influence the firm's payoff distribution by costly monitoring. Risk aversion gives him an incentive to sell some shares to a finite number of strategic outside investors, but selling shares reduces the owner's incentive to monitor. The model solves the owner's trade-off between monitoring and risk sharing. The moral hazard problem between the players leads to less monitoring than at the social optimum. Also, there is inefficient risk sharing in equilibrium. Surprisingly, in the special case with a single outside investor, the monitoring technology can make both players worse off. The reason is that the owner and the investor are strategic, which leads to extra distortions in the form of less risk sharing. The fraction of the firm sold is a non-monotonic function of the cost of monitoring, which contradicts traditional models of monitoring. If all bargaining power is with the owner, risk sharing improves but the parameter region where monitoring is feasible contracts. The same effect is observable if the number of outside investors increases, because of a free-rider problem between the investors.

1 Introduction

Casual observation suggests that some stock market investors are more sophisticated than others. Some investors can influence the operations of a firm in a way to increase efficiency, and thereby increase the value of the firm. Let us call this ability to increase firm value a "monitoring technology" of the investor. The problem is, however, that using such a technology comes with a cost. A large investor has to gather information and attend annual meetings to detect any malfunctioning and to implement changes in corporate strategy. Any "monitoring" by such an investor is a public good and is accompanied by a free-rider problem: All shareholders benefit from an increase in firm value, but only the monitoring investor carries the cost. Several authors have pointed out a trade-off for shareholders with a monitoring technology: If he owns a larger fraction of the firm, he has stronger incentives to engage in monitoring. On the other hand, owning a larger fraction means that the shareholder is less diversified and bears more of the idiosyncratic risk of the particular firm.

This paper proposes a simple model to analyze this trade-off between monitoring and risk sharing. Assume that a risk averse firm owner holds all shares in a company and is

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endowed with a monitoring technology that allows him to increase the value of the firm at a certain private cost. Risk aversion gives him an incentive to sell some of his shares, but at the same time he realizes that selling shares will reduce his incentives to monitor. It is assumed that there is a single outside investor, who may buy shares from the firm owner. Thus, there are two players engaged in a non-cooperative game. The solution of the game shows that in equilibrium there is less monitoring than in a model where a social planner maximizes total welfare. The second result is that the moral hazard problem distorts risk sharing between the firm owner and the investor. As a consequence, the two players never reach the total welfare that could be reached by a social planner. The model also yields some surprising predictions. For instance, intuition might suggest that if one agent is endowed with a monitoring technology that allows him to increase firm value, then at least one of the two agents will profit from this technology. Interestingly, this is not the case in the current model. It turns out that for some parameter values both players are worse off by the availability of a monitoring technology. This is because in some cases risk sharing is strongly distorted, which reduces the welfare of both players. The model is then extended to multiple strategic investors. It is shown that the region of parameter values the supports monitoring becomes smaller as the number of investors grows. Also, the owner is better off when he can sell shares to mutiple investors. For some parameter values he sells the whole firm, which is equivalent to perfect risk sharing.

The main innovation in this paper is the assumption that the outside investor is strategic, as traditional models assume price-taking investors. Also, the surprising result that for some parameter values both players are worse off is new. Further the present framework allows to analyze what happens when the number of outside investors grows. This was not possible in traditional models because they essentially assumed a continuum of investors.

There are several papers in the literature that address the same issues as here. The moral hazard problems that arise if a firm owner sells a fraction of his firm to outside investors were introduced by Jensen and Meckling (1976). They study the case when a firm owner sells a fraction of his firm and subsequently consumes more non-pecuniary benefits than before. Closer to the present model, however, is Admati, Pfleiderer, and Zechner (1994). The authors of this paper explicitly model the trade-off between monitoring and risk sharing of a large shareholder, i.e. a shareholder who does not necessarily own the whole firm. An important difference to the present model is that they assume a continuum of price-taking outside investors, whereas here a single strategic outside investor is considered. The one-period version of the model of Admati, Pfleiderer, and Zechner (1994) yields similar results to the present analysis, namely that the large shareholder will trade off monitoring against risk sharing and that there will be less risk sharing than at a social optimum. Huddart (1993) is related in the sense that the author models the trade-off between risk-sharing and monitoring. His model additionally incorporates the optimal compensation contract of a firm manager, but again all outside investors are non-strategic.

Other papers focus on different problems related to shareholder monitoring. Shleifer and Vishny (1986) analyze the role of a large shareholder in takeovers. They show that a takeover is more likely to be successful if the raider is a large shareholder in the firm. Burkart, Gromb, and Panunzi (1997) argue that a less concentrated ownership structure might also have its benefits: Large shareholders, as monitors, might destroy the incentives of firm managers to show initiative, i.e. to invest in firm-specific human capital. Maug (1998) analyzes whether a liquid stock market strengthens or weakens the incentives of a large shareholder to engage in monitoring. Stoughton and Zechner (1998) investigate different IPO mechanisms together with the presence of a large institutional investor who has the ability to increase firm value by monitoring. In their paper the buyer of shares is endowed with the monitoring technology, whereas in the present model it is the seller of the shares. Lastly, in a recent contribution, DeMarzo and Urosevic (2009) extend the model of Admati, Pfleiderer, and Zechner (1994) to the infinite time horizon setting with many trading rounds. Again, the outside investors in their model are price-taking, hence the strategic interaction between the large shareholder and the investors is neglected.

2 The Model

The model is a non-cooperative game with perfect information and two players. The first player is the firm owner denoted O, who initially owns all shares in a firm. The second player is an outside investor called I, who can acquire some shares from the owner. The firm is modeled abstractly as a normally distributed random payoff D, with

$$D \sim N(\mu, \sigma^2), \quad \text{and} \quad \mu, \sigma > 0.$$

The firm owner has a monitoring technology by which he can increase the mean of the random payoff. If he decides to monitor, the distribution changes to

$$D' \sim N(\mu + m, \sigma^2), \quad \text{with} \quad m > 0.$$

Monitoring is costly: If the owner decides to monitor, he has to pay $c \in (0, m)$ units of consumption. The total endowment of the owner thus amounts to his shares in the firm and his monitoring technology. The outside investor, on the other hand, is endowed with an initial wealth of $W > 0$ units of consumption.

The timeline of events is as follows: First, the owner selects a price $p \geq 0$ at which he offers his shares for sale. Then the investor chooses how many shares $x \in [0, 1]$ he wants to acquire, where the total number of shares available is 1. Following the investor's move, the owner decides whether he wants to use his monitoring technology, which is a binary decision. The variable $y \in \{0, 1\}$ indicates the monitoring decision of the owner, with $y = 1$ standing for monitoring. The part of the players' wealth that is not held in stocks is invested in a riskfree asset with constant return $r = 0$. Finally, payoffs realize and the costs of monitoring are paid. Figure 1 summarizes the timeline of events.

Both players maximize expected utility of terminal wealth, W_T, and their Bernoulli utility function has constant absolute risk aversion (CARA):

$$U(W_T) = E[u(W_T)] = E\left[-e^{-aW_T}\right].$$

The constant $a > 0$ denotes the coefficient of absolute risk aversion and for simplicity is assumed to be the same for both players. Note that terminal wealth is a random variable that

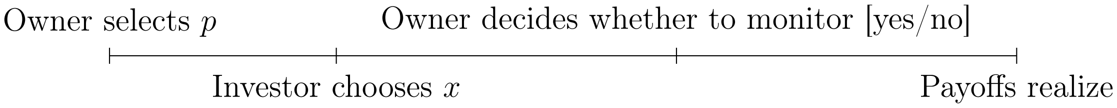

Figure 1: Timeline of events in the model.

depends on the stock price p, the amount invested in the firm and the monitoring decision. Additionally, terminal wealth is normally distributed. Due to the normal distribution and the special form of the utility function, maximizing the owner's utility above is equivalent to maximizing

$$U_O(p, x, y) = xp - yc + (1 - x)(\mu + ym) - \frac{a\sigma^2}{2}(1 - x)^2. \tag{1}$$

Analogously, maximizing the outside investor's utility is equivalent to maximizing

$$U_I(p, x, y) = W + x(\mu + ym - p) - \frac{a\sigma^2}{2}x^2. \tag{2}$$

2.1 Solution of the model by backwards induction

Backwards induction is used to find a subgame-perfect Nash equilibrium of the game. First, the monitoring decision of the owner O is solved, given a price p and a fraction of the firm x that is acquired by the investor I. Then the optimal demand for stocks x of the investor is calculated, given some p, and finally the stock price p that is optimally chosen by the owner is derived.

The first step is to find out under which conditions the owner will use his monitoring technology. Monitoring is optimal for the owner if his utility with monitoring is at least as large as his utility without monitoring. This is formally equivalent to the following inequality:

$$xp - c + (1 - x)(\mu + m) - \frac{a\sigma^2}{2}(1 - x)^2 \geq xp + (1 - x)\mu - \frac{a\sigma^2}{2}(1 - x)^2,$$

which can be reduced to $(1 - x)m \geq c$. This simply says that the owner's share of the benefits from monitoring are at least as large as his costs of monitoring. More precisely, if $(1-x)m > c$, then monitoring is optimal. Analogously, if $(1-x)m < c$, then not monitoring is optimal. At $(1-x)m = c$ the owner is indifferent between monitoring and not monitoring.

Now turn to deriving the investor's demand for stocks. Note that, for any given p, I chooses $x \in [0, 1]$ to maximize his utility given in equation (2). However, the number of shares x acquired directly influences the owner's monitoring decision. For x below the threshold $1 - c/m$, the investor knows that there will be monitoring, and for x above this threshold there will be no monitoring. Therefore, the investor's utility function exhibits a jump at $x = 1 - \frac{c}{m}$. This can be seen in Figure 2, where the solid parabolae represent the investor's utility if he expects monitoring, and the dashed parabolae depict his utility if he expects no monitoring.

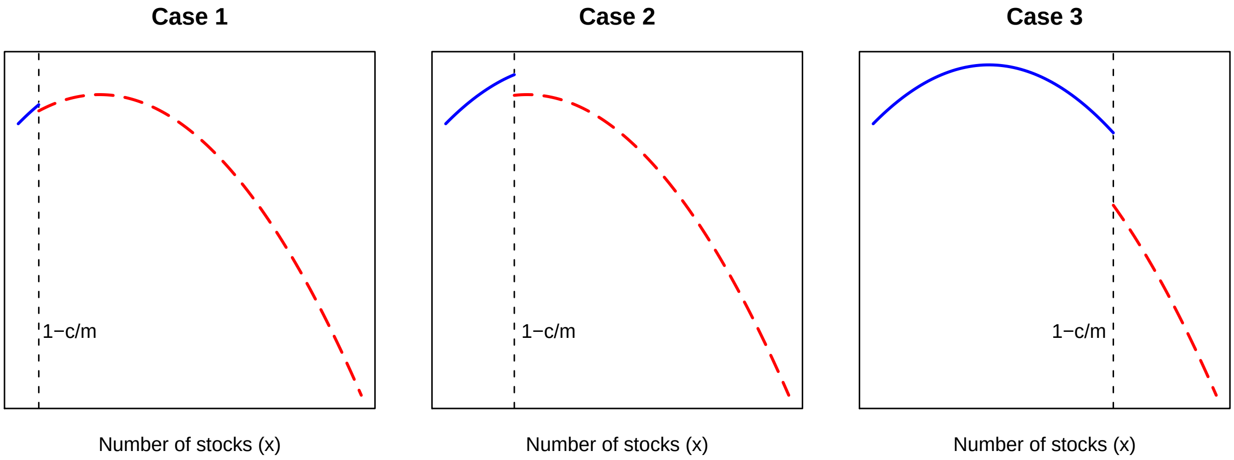

Figure 2: These figures show the utility of investor I as a function of the number of stocks x he acquires. In each case, the solid parabola depicts his utility if the owner monitors, whereas the dashed parabola depicts his utility when the owner does not monitor.

In Figure 2 there are three cases. This is so because for different values of p, the positions of the parabolae change relative to the constant $1 - \frac{c}{m}$. In the first case, p is such that the maximum of I's utility is at the critical point of the dashed parabola. Case 2, however, differs because the maximum of I's utility is at $x = 1 - \frac{c}{m}$. Finally, in the third case the investor's utility is maximized at the critical point of the solid parabola. Thus, for each of the three cases, the investor exhibits a different demand function. Figure 2 is a stylized picture with the purpose of introducing the logic of the argument. Nevertheless, these informal statements can be made precise and the three demand functions can be derived analytically. The following Lemma summarizes the demand functions for all cases.

Lemma 1. *The investor's demand for shares x as a function of the price p is piecewise linear of the form*

$$x(p) = \begin{cases} \frac{\mu - p}{a\sigma^2} & \text{if } p \leq \underline{p}, \\ 1 - \frac{c}{m} & \text{if } \underline{p} \leq p \leq \bar{p}, \\ \frac{\mu + m - p}{a\sigma^2} & \text{if } \bar{p} \leq p, \end{cases}$$

where $\underline{p} = \mu - a\sigma^2(1 - \frac{c}{m}) - \sqrt{2a\sigma^2(m-c)}$ *and* $\bar{p} = \mu + m - a\sigma^2(1 - \frac{c}{m})$, *with a discontinuity at* $p = \underline{p}$.

Proof. The structure of the proof is the following. First the two thresholds $\underline{p}$ and $\bar{p}$ are calculated. Then the demand function for the three regions defined by the two thresholds is derived.

Take equation (2) and insert $y = 0$ and $y = 1$. This yields the two parabolae that can be seen in each particular case of Figure 2. Denote these parabolae $U_I(p, x, 0)$ and $U_I(p, x, 1)$. Note that the parabolae are shifted down and to the left if p increases. In other words, if p is small, then the investor's utility function is similar to case 1 of Figure 2. Then, as p increases, a switch to case 2 occurs at some point. If p increases even further, there is a transition to case 3. The threshold for the change from case 1 to case 2 is given implicitly by

$$U_I(p, 1 - c/m, 1) = \max_x U_I(p, x, 0).$$

This is a quadratic equation in p that yields the first threshold

$$\underline{p} = \mu - a\sigma^2(1 - \frac{c}{m}) - \sqrt{2a\sigma^2 m(1 - \frac{c}{m})}.$$

Note that only the smaller solution of the quadratic equation is relevant because the maximizer of $U_I(p, x, 1)$ is larger than the maximizer of $U_I(p, x, 0)$ (see case 1 in Figure 2). The second threshold is given implicitly by the equation

$$\arg\max_x U_I(p, x, 1) = 1 - \frac{c}{m},$$

which yields $\bar{p} = \mu + m - a\sigma^2(1 - \frac{c}{m})$.

Next derive the demand functions for the three regions $p \leq \underline{p}$, $\underline{p} \leq p \leq \bar{p}$, and $\bar{p} \leq p$. In the first region, which corresponds to case 1, finding the optimal demand x is equivalent to maximizing the dashed parabola defined by $U_I(p, x, 0)$ with respect to x, which yields $x = \frac{\mu - p}{a\sigma^2}$. In the second region, demand is constant at $x = 1 - \frac{c}{m}$. Lastly, in region 3 (case 3) the optimal demand is given by maximizing the solid parabola $U_I(p, x, 1)$ with respect to x. This yields $x = \frac{\mu + m - p}{a\sigma^2}$, and the derivation of the piecewise linear demand function is complete. ☐

Figure 3 shows the demand function in Lemma 1 in stylized form. It also incorporates the monitoring decision of the owner. For low prices demand is high and the owner subsequently owns a small number of shares. This reduces his incentives to monitor, and if $p \leq \underline{p}$ it is optimal for him not to monitor. On the other hand, if the stock price is high, $\bar{p} \leq p$, then the investor buys a small number of shares, leaving enough shares to the owner so that it is optimal for the latter to monitor. Interestingly, there is a third case where $\underline{p} \leq p \leq \bar{p}$ and demand $x(p)$ is constant. In this region the investor buys just enough shares to make the owner indifferent between monitoring and not monitoring.

Finally, turn to the optimal price p as chosen by the firm owner. For that purpose, look at the three segments of the demand function in Lemma 1 separately. Consider the first segment, where demand takes the form $x(p) = \frac{\mu - p}{a\sigma^2}$. Suppose the owner takes this demand function as given and chooses p so as to maximize his expected utility given by equation (1), with $y = 0$. Together with $p \leq \underline{p}$ this means the owner solves a constrained optimization problem. The same argument is used for the second segment of the demand function. Again, the owner takes the demand $x(p) = 1 - \frac{c}{m}$ as given and chooses a price to maximize his expected utility. Since in this case demand is constant, the owner chooses a price that is as large as possible. His constraint is given by $\underline{p} \leq p \leq \bar{p}$, so he selects $p^* = \bar{p}$. Finally, consider the third segment of the investor's demand function. The owner maximizes equation (1), with $y = 1$, $x(p) = \frac{\mu + m - p}{a\sigma^2}$, and $\bar{p} \leq p$. This informal case-by-case analysis can be made precise, and the results are summarized by the next Lemma.

Lemma 2. *The optimal price p^* is derived for each segment of the investor's demand function given in Lemma 1.*

1. *In the first segment, where $x(p) = \frac{\mu - p}{a\sigma^2}$, the optimal price is*

$$p^* = \min\{\mu - \frac{a\sigma^2}{3}, \underline{p}\}, \quad \textit{where} \quad c \geq \frac{2a\sigma^2 m - 3m^2 + \sqrt{3(2a\sigma^2 m^3 + 3m^4)}}{3a\sigma^2} \equiv \bar{c}$$

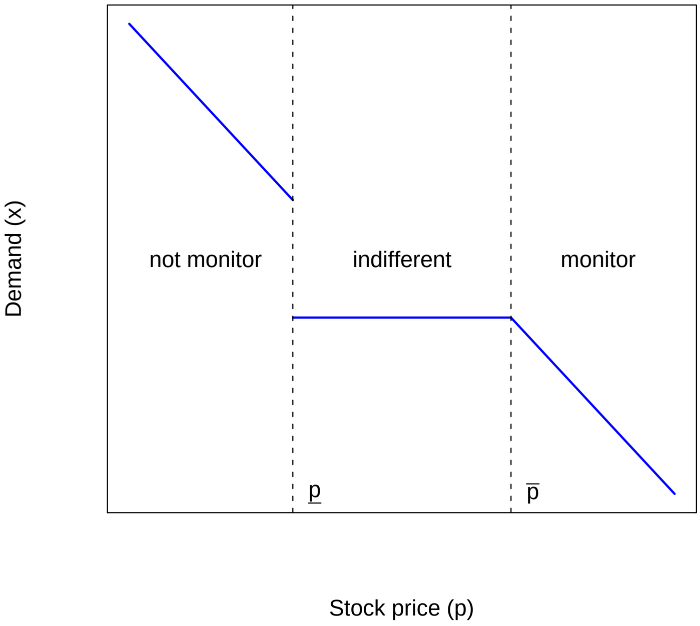

Figure 3: The investor's demand for stocks x as a function of the stock price p. The two vertical dashed lines depict the thresholds $\underline{p}$ and $\bar{p}$, respectively.

has to hold for an interior optimum.

2. *In the second segment, where demand is constant at* $x(p) = 1 - c/m$, *the optimal price is*
$$p^* = \bar{p} = \mu + m - a\sigma^2(1 - \frac{c}{m}).$$

3. *In the third segment, where* $x(p) = \frac{\mu+m-p}{a\sigma^2}$, *the optimal price is*
$$p^* = \max\{\mu + m - \frac{a\sigma^2}{3}, \bar{p}\},$$
where $c \leq \frac{2}{3}m$ *has to hold for an interior optimum.*

Proof. For the first part of the Lemma, take the owner's utility function in equation (1), insert $y = 0$ and $x = \frac{\mu-p}{a\sigma^2}$, and maximize with respect to p, constrained by $p \leq \underline{p}$. This yields $p^* = \mu - \frac{a\sigma^2}{3}$ as an interior solution. For the interior solution the constraint is only satisfied if $c \geq \frac{2a\sigma^2 m - 3m^2 + \sqrt{3(2a\sigma^2 m^3 + 3m^4)}}{3a\sigma^2}$. The optimal price is thus $p^* = \min\{\mu - \frac{a\sigma^2}{3}, \underline{p}\}$.

The second part of the proof is analogous. Insert $y = 1$ (or, by indifference, $y = 0$) and $x = 1 - c/m$ to the owner's utility function in equation (1). The owner's utility is increasing in p, so he optimally chooses a price that is as large as possible. By Lemma 1, the constraint for the second segment of the investor's demand function is $\underline{p} \leq p \leq \bar{p}$, so the owner chooses

$p^* = \bar{p}$.

For the third part of the proof, insert $y = 1$ and $x = \frac{\mu+m-p}{a\sigma^2}$ to the owner's utility function in equation (1). Then maximize this function with respect to p, constrained by $\bar{p} \leq p$. The interior solution of this problem is $p^* = \mu + m - \frac{a\sigma^2}{3}$. Note that the interior solution is only feasible if $c \leq \frac{2}{3}m$. The optimal price is thus $p^* = \max\{\mu + m - \frac{a\sigma^2}{3}, \bar{p}\}$. $\square$

Note that the prices in Lemma 2 are only optimal from a local perspective: For each particular segment of the demand function, the owner chooses a price that is optimal within that segment. What is missing is a global solution to the owner's optimization problem. For that purpose, proceed as follows: Calculate the owner's indirect utility for each price in Lemma 2 separately, and then compare these indirect utilities. It turns out that there are three different types of global optima, corresponding to three types of subgame-perfect Nash equilibria of the game, depending on the exogenous parameters c, m, a and σ. The following Proposition puts together and summarizes the parameter regions that determine the particular type of equilibrium.

Proposition 1. *Depending on parameters, the following are subgame-perfect Nash equilibria.*

1. *If $\hat{c} \leq c < m$, with $\hat{c} = \frac{2(3a\sigma^2 m - 2m^2 + m\sqrt{6a\sigma^2 m + 4m^2})}{9a\sigma^2}$, then $p = \min\{\mu - \frac{a\sigma^2}{3}, \underline{p}\}$, where $\underline{p} = \mu - a\sigma^2(1 - \frac{c}{m}) - \sqrt{2a\sigma^2(m-c)}$, $x = \frac{\mu - p}{a\sigma^2}$, and the owner does not monitor, i.e. $y = 0$.*
2. *If $\frac{2}{3}m \leq c \leq \hat{c}$, then $p = \mu + m - a\sigma^2(1 - \frac{c}{m})$, $x = 1 - c/m$, and the owner monitors, i.e. $y = 1$.*
3. *If $0 < c \leq \frac{2}{3}m$, then $p = \mu + m - \frac{a\sigma^2}{3}$, $x = \frac{1}{3}$, and the owner monitors, i.e. $y = 1$.*

Proof. The objective is to find the optimal price p for the owner. Note that Lemma 2 provides three regions with separate, locally optimal prices. The plan is to calculate the indirect utility for each of the three prices in Lemma 2, and check, for different parameter values, which of the three cases is optimal from the owner's perspective.

To calculate the utility for the interior optimum of the first case in Lemma 2, insert $p = \mu - \frac{a\sigma^2}{3}$, $x = 1/3$, and $y = 0$ into (1), which yields $\mu - \frac{a\sigma^2}{3}$. For the corner solution, insert $p = \underline{p}$, $x = \frac{\mu - p}{a\sigma^2}$, and $y = 0$. Similarly, the indirect utility for the second case in Lemma 2 is

$$U_O(\bar{p}, 1 - c/m, 1) = \mu + m - c + a\sigma^2(2c/m - 1) - \frac{3a\sigma^2}{2}(c/m)^2.$$

The indirect utility for the interior optimum of the third case in Lemma 2 is

$$U_O(\mu + m - \frac{a\sigma^2}{3}, 1/3, 1) = \mu + m - c - \frac{a\sigma^2}{3},$$

while the utility at the corner solution it is given by

$$U_O(\bar{p}, \frac{\mu + m - \bar{p}}{a\sigma^2}, 1), \quad \text{with} \quad \bar{p} = \mu + m - a\sigma^2(1 - \frac{c}{m}).$$

Comparing case 2 to the interior optimum of case 3 reveals that case 3 is always better in terms of indirect utility, provided that the interior optimum is feasible, i.e. $c \leq \frac{2}{3}m$. It is not necessary to compare case 2 to the boundary solution of case 3, as these are identical to the owner in terms of his utility.

Now compare the interior optimum of case 1 to case 2. Formally, the latter is preferred to the former if the following inequality holds:

$$\mu + m - c + a\sigma^2(2c/m - 1) - \frac{3a\sigma^2}{2}(c/m)^2 \geq \mu - \frac{a\sigma^2}{3},$$

which can be reformulated by solving a quadratic equation as

$$c \leq \frac{2a\sigma^2 m - m^2 + \sqrt{2a\sigma^2 m^3 + m^4}}{3a\sigma^2}.$$

In other words, there is an upper bound on c such that below this bound, case 2 is preferred to the interior optimum of case 1 by the owner. But by Lemma 2 there is a feasibility bound on the interior solution of case 1, which is a lower bound on c. The final question is then to determine whether the upper bound on c just derived is smaller than the lower bound on c from Lemma 2, i.e.

$$\frac{2a\sigma^2 m - m^2 + \sqrt{2a\sigma^2 m^3 + m^4}}{3a\sigma^2} \leq \frac{2a\sigma^2 m - 3m^2 + \sqrt{3(2a\sigma^2 m^3 + 3m^4)}}{3a\sigma^2}.$$

It turns out that this is indeed the case, and that the inequality is strict. To see this, simplify the inequality to

$$2m + \sqrt{2a\sigma^2 m + m^2} \leq \sqrt{6a\sigma^2 m + 9m^2}.$$

Square both sides and simplify further to

$$\sqrt{2a\sigma^2 m + m^2} \leq m + a\sigma^2.$$

Finally, square both sides again and reduce the inequality to

$$0 \leq (a\sigma^2)^2.$$

The RHS is strictly positive because of the assumptions $a > 0$ and $\sigma^2 > 0$. Thus, in terms of Figure 4 there is a threshold strictly smaller than $\bar{c}$ such that the owner prefers the interior optimum of case 1 to case 2 above that threshold. However, the feasibility condition says that the interior optimum of case 1 is only attainable for $c \geq \bar{c}$, which is a stronger constraint on c.

Now compare the corner solution of case 1 to case 2. The former is preferred to the latter if the following inequality holds:

$$U_O(\underline{p}, \frac{\mu - \underline{p}}{a\sigma^2}, 0) \geq U_O(\bar{p}, 1 - c/m, 1).$$

After substituting for $\underline{p}$ and $\bar{p}$, this can be rearranged and simplified to

$$9a\sigma^2(c/m)^2 - 12a\sigma^2 c/m + 4a\sigma^2 - 8(m - c) \geq 0.$$

When simplifying the inequality to the given form, the assumption $c \geq 2/3m$ is required. This is not a problem because if c is smaller than $2/3m$, then case 3 dominates all other cases, which becomes clear at the end of the proof. Take c as a variable, then the inequality has two roots:

$$c = \frac{2(3a\sigma^2 m - 2m^2 \pm m\sqrt{6a\sigma^2 m + 4m^2})}{9a\sigma^2}.$$

One can easily check that the smaller root is below $2/3m$, while the larger root is above $2/3m$. By the same argument as before, only the larger root is relevant. The corner solution of case 1 is superior to case 2 if c is above the larger root. To distinguish this threshold from $\bar{c}$, denote it $\hat{c}$:

$$c \geq \frac{2(3a\sigma^2 m - 2m^2 + m\sqrt{6a\sigma^2 m + 4m^2})}{9a\sigma^2} \equiv \hat{c}.$$

Finally, the question is whether $\hat{c} < \bar{c}$. Write

$$\frac{2(3a\sigma^2 m - 2m^2 + m\sqrt{6a\sigma^2 m + 4m^2})}{9a\sigma^2} \leq \frac{2a\sigma^2 m - 3m^2 + \sqrt{3(2a\sigma^2 m^3 + 3m^4)}}{3a\sigma^2},$$

then square both sides twice and simplify to

$$0 \leq 9(a\sigma^2)^2.$$

This inequality strictly holds by the assumptions $a > 0$ and $\sigma > 0$. Hence, $\hat{c}$ is always below $\bar{c}$. To summarize, if $\bar{c} \leq c$, then the interior optimum of case 1 is superior to case 2, and over the region $\hat{c} \leq c \leq \bar{c}$ the corner solution of case 1 is preferred to case 2. This completes the comparison of case 1 and case 2.

Note that the interior optima of case 1 and case 3 cannot be contrasted because their feasible regions do not intersect. Additionally, their respective corner solutions do not have to be compared as this would be formally equivalent to comparing the corner solution of case 1 to case 2, which has already been done. To summarize, the owner chooses a price according to the interior optimum of case 3 in the region $c \leq \frac{2}{3}m$, as this is preferred to all other cases. On the other side of the spectrum of Figure 4, the owner chooses case 1 in the region $\hat{c} \leq c$, because he prefers this to case 2. To be more precise, the owner implements the interior optimum of case 1 in the region $\bar{c} \leq c$, while he uses the corner solution over the interval $\hat{c} \leq c \leq \bar{c}$. Finally, over the remaining region $\frac{2}{3}m \leq c \leq \hat{c}$, the owner chooses a price according to the solution of case 2. □

It is important to understand that in the region $2/3m \leq c \leq \hat{c}$ the owner is indifferent between monitoring and not monitoring, but in a subgame-perfect Nash equilibrium he monitors. There is an intuitive argument for this: If he did not monitor, the investor would have an incentive to demand a slightly smaller number of stocks. This would make it optimal for the owner to monitor. Hence, not monitoring over the region $2/3m \leq c \leq \hat{c}$ cannot occur in equilibrium.

Focussing on two exogenous parameters of the model, namely monitoring costs c and the expected benefits from monitoring m, the three regions can be depicted graphically (for fixed a and σ) as in Figure 4. It shows that if c is relatively small compared to m, then the equilibrium is of type 3 in Proposition 1. This region is depicted by the dashed line. On the other extreme, if c is large relative to m, then type 1 of the equilibria of Proposition 1 occurs, as shown by the dotted line. The intermediate case is depicted by the solid line: It represents the type 2 equilibrium in Proposition 1, for intermediate values of c.

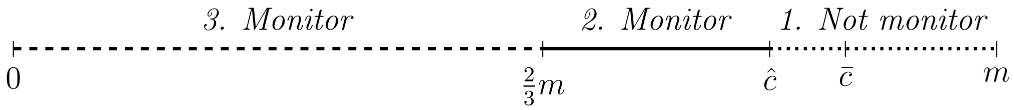

Figure 4: Depending of the values of c and m (for fixed a and σ), three different types of subgame-perfect Nash equilibria exist. The parameter that varies in this graph is c. The numbers 1, 2, and 3 refer to the numbering in Proposition 1. The constants $\hat{c}$ and $\bar{c}$ are defined in Proposition 1 and Lemma 2, respectively.

Note that there is a region $\hat{c} \leq c < m$, where monitoring would be optimal from the perspective of a social planner who maximizes total welfare, but in equilibrium there is no monitoring. The reason for this is the moral hazard problem. If the owner would be able to commit to monitoring, then the problem could be resolved.

Another interesting result is that the region of inefficient monitoring, $\hat{c} \leq c < m$, is increasing in a and in σ. In other words, if risk aversion or firm risk increases, the region of inefficient monitoring expands. To see this, take $a\sigma^2$ as one variable. This is without loss of generality because in all results of the model, risk aversion and risk show up as one term. Then check how $\hat{c}$ changes when $a\sigma^2$ increases, which yields

$$\frac{\partial \hat{c}}{\partial a\sigma^2} = \frac{\partial}{\partial a\sigma^2}\left[\frac{2(3a\sigma^2 m - 2m^2 + m\sqrt{6a\sigma^2 m + 4m^2})}{9a\sigma^2}\right] < 0.$$

It is also interesting that both for small c (relative to m) and large c, the fraction of the firm that is sold to the investor is the same. Intuition might suggest that the owner sells a lower fraction if c is small and a higher fraction if c is large, but this is not the case.

In the two intermediate regions, $\frac{2}{3}m \leq c \leq \hat{c}$ and $\hat{c} \leq c \leq \bar{c}$, the fraction of the firm sold to the investor is decreasing in c. This, together with the previous observation, yields another surprising result: the fraction of the firm sold is non-monotonic and discontinuous in c (for fixed m, a and σ). The pattern of x for varying c is different from the distribution of shares that a social planner would impose. In that case, x would be constant over all c at $x = \frac{1}{2}$ (see Figure 5). This leads to the next subsection, the welfare analysis and the distribution of total welfare among the two players.

2.2 Welfare analysis

Based on the results in Proposition 1 one can calculate the indirect utilities of both players and the distribution of total welfare among them. Total welfare is given by

$$U_O(p,x,y) + U_I(p,x,y) = W + \mu + y(m-c) - a\sigma^2(x^2 - x + \frac{1}{2}).$$

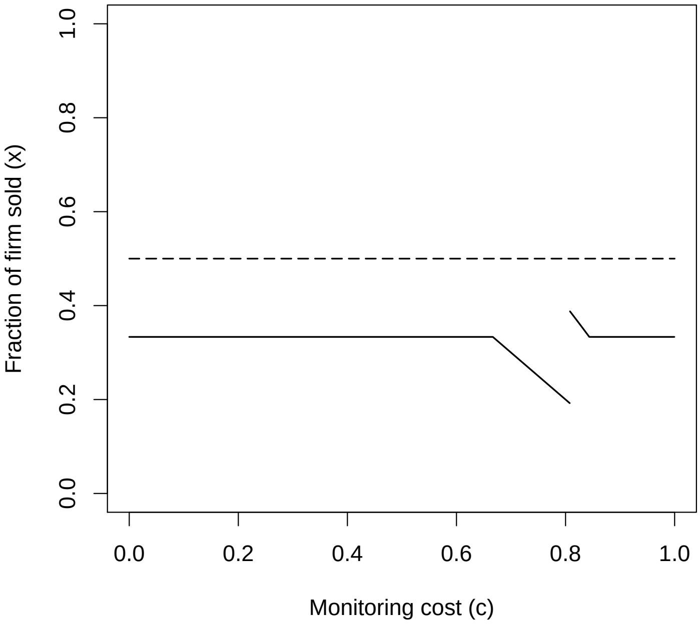

Figure 5: The solid line shows the fraction of the firm sold to the investor, x, for different values of c. The longdashed line depicts the optimal sharing rule from the perspective of a social planner. The graph is based on the parameter values $\sigma^2 = m = W = 1$, $\mu = 10$, $a = 10$.

Figure 6 shows the indirect utility of the owner (dashed line), of the investor (dotted line), and total welfare (solid line). The longdashed line on the top shows total welfare in a hypothetical economy where a social planner maximizes the sum of the two players' utilities. The picture reveals that the players never achieve the same total welfare as the social planner would achieve. Second, both the owner's welfare and total welfare as functions of c are non-monotonic and discontinuous. Another observation is that the deviation from the social optimum is largest for case 2 in Proposition 1. Especially when c is slightly below the threshold $\hat{c}$, the model yields an equilibrium where welfare is significantly smaller than potential welfare. The reason for this is that in such equilibria, as can be seen in Figure 5, risk sharing between the two players deteriorates. Under optimal risk sharing, the players would share the firm in equal proportions, but in the region $\frac{2}{3}m \leq c \leq \hat{c}$ distortions from this first-best optimum are quite large.

However, the most striking observation in Figure 6 is that at the point $\lim_{c \nearrow \hat{c}}$, i.e. c is slightly below $\hat{c}$, welfare is below the level that would be achieved if there were no monitoring technology. Actually the figure reveals that if c is slightly below $\hat{c}$ then *both* players are worse off compared to a model without a monitoring technology. To see the players' utility levels without a monitoring technology, take the horizontal line segments (dashed or dotted) for c close to 1, and stretch them over the full interval $(0, 1)$. Then it becomes clear that for c slightly below $\hat{c}$, the equilibrium utilities of the players are below the utilities without monitoring technology. This is because risk sharing is distorted so much that both players incur a loss that is not outweighed by the benefits of monitoring.

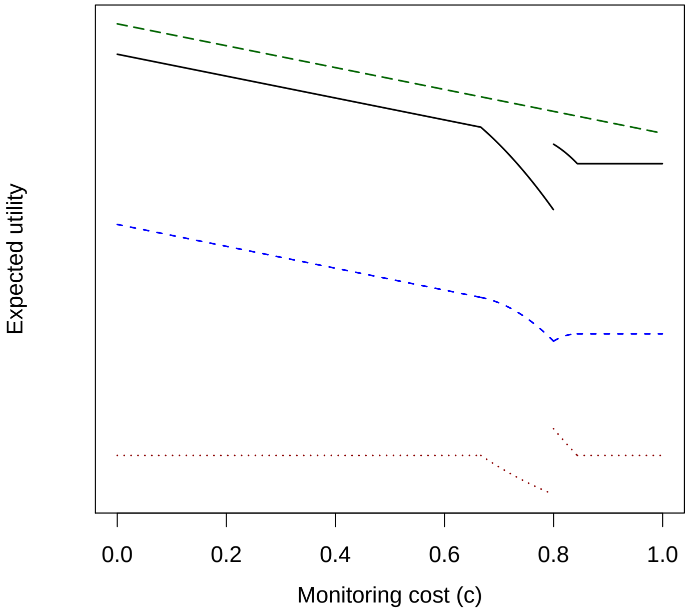

Figure 6: Utility of the owner (dashed line), utility of the investor (dotted line), and total welfare (solid line). The longdashed line on the top shows total welfare in a hypothetical economy where a social planner maximizes the sum of the two players' utilities. The graph is based on the parameter values $\sigma^2 = m = W = 1$, $\mu = 10$, $a = 10$.

3 Model with two investors

This section extends the basic model by adding a second outside investor. The two investors, labeled i and j, are assumed to have identical preferences and endowments. Investor i chooses x_i to maximize

$$U_i(p, x_i, y) = \frac{W}{2} + x_i(\mu + ym - p) - \frac{a\sigma^2}{2} x_i^2. \tag{3}$$

Compared to the previous model, the initial wealth of each investor is reduced to $\frac{W}{2}$. This is in order to make the welfare analyses of the two models comparable. Also, the total number of shares must not exceed total supply,

$$x \equiv x_i + x_j \leq 1, \text{ with } x_i, x_j \geq 0.$$

The sequence of moves in the game is now as follows. First, the owner O chooses a price p. Then investors i and j simultaneously decide how many stocks they purchase, x_i and x_j, respectively. Finally, the owner makes a decision whether to monitor, and payoffs realize.

Backwards induction is used to find the subgame-perfect Nash equilibria of the game. Therefore the monitoring decision of the owner has to be solved first. This is followed by the optimal demand of investors i and j, and lastly the optimal price as chosen by the owner.

The optimal monitoring decision follows by straightforward extension of the previous model:

$$y = \begin{cases} 1 & \text{if } (1 - x_i - x_j)m > c \\ 0 & \text{if } (1 - x_i - x_j)m < c. \end{cases}$$

At $(1 - x_i - x_j)m = c$ the owner is indifferent between monitoring and not monitoring.

The optimal monitoring decision is now used to derive the investors' demand for stocks. Figure 7 shows the utility function of investor i, together with the optimal monitoring decision of the owner.

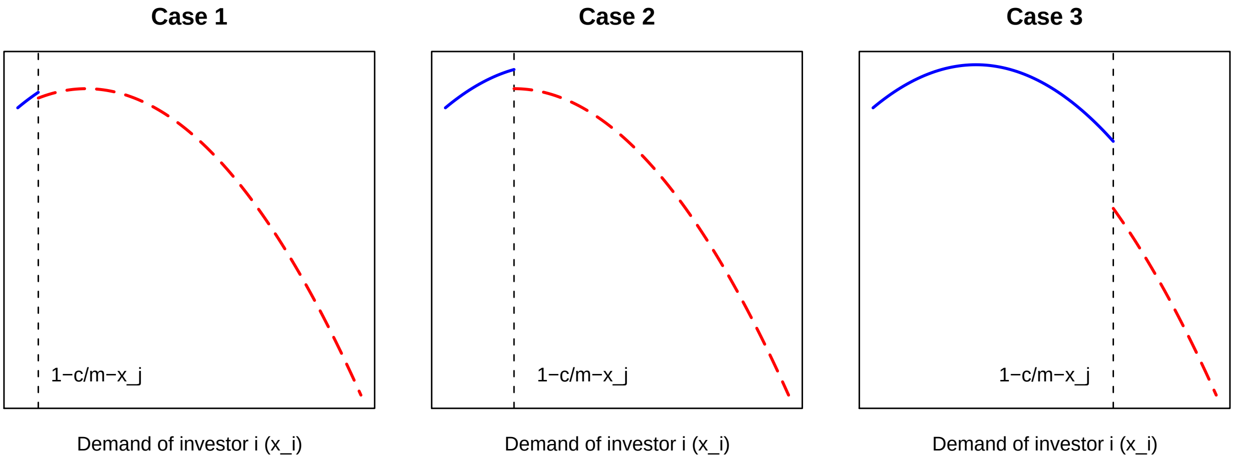

Figure 7: Utility of investor i as a function of the number of stocks x_i he acquires. In each case, the solid parabola depicts his utility if the owner monitors, whereas the dashed parabola depicts his utility when the owner does not monitor.

Figure 7 reveals how the optimal demand of investor i depends on the demand of investor j, for a fixed price p. Since the two investors are assumed to choose their demands simultaneously, the Nash equilibria of the game between them have to be found in order to determine their demand functions. As suggested by Figure 7, the best reply correspondence of each player breaks down to three regions.

$$x_i = \begin{cases} \frac{\mu - p}{a\sigma^2} & \text{if } U_i(p, 1 - c/m - x_j, 1) \leq \max_{x_i} U_i(p, x_i, 0), \\ 1 - c/m - x_j & \text{if } U_i(p, 1 - c/m - x_j, 1) \geq \max_{x_i} U_i(p, x_i, 0) \\ & \text{and } 1 - c/m - x_j \leq \arg\max_{x_i} U_i(p, x_i, 1), \\ \frac{\mu + m - p}{a\sigma^2} & \text{else.} \end{cases}$$

After substituting from (3) this best reply correspondence can be simplified to

$$x_i = \begin{cases} \frac{\mu - p}{a\sigma^2} & \text{if } (\mu + m - p)(1 - c/m - x_j) - \frac{a\sigma^2}{2}(1 - c/m - x_j)^2 \leq \frac{(\mu - p)^2}{2a\sigma^2}, \\ 1 - c/m - x_j & \text{if } (\mu + m - p)(1 - c/m - x_j) - \frac{a\sigma^2}{2}(1 - c/m - x_j)^2 \geq \frac{(\mu - p)^2}{2a\sigma^2} \\ & \text{and } 1 - c/m - x_j \leq \frac{\mu + m - p}{a\sigma^2}, \\ \frac{\mu + m - p}{a\sigma^2} & \text{else.} \end{cases} \quad (4)$$

Figure 8 depicts the best reply correspondences of the two investors. The graph shows that each investor's best reply correspondence can be approximated by a function from $[0,1]$ to $[0,1]$ with a single discontinuity, which is an upward jump. This approximation always has a fixed point which is not at the discontinuity. Hence, both best reply correspondences have a fixed point, which implies that the game always has a *symmetric* Nash equilibrium in pure strategies. Moreover, for some parameter values there is a continuum of asymmetric pure strategy Nash equilibria, as shown in Figure 8.

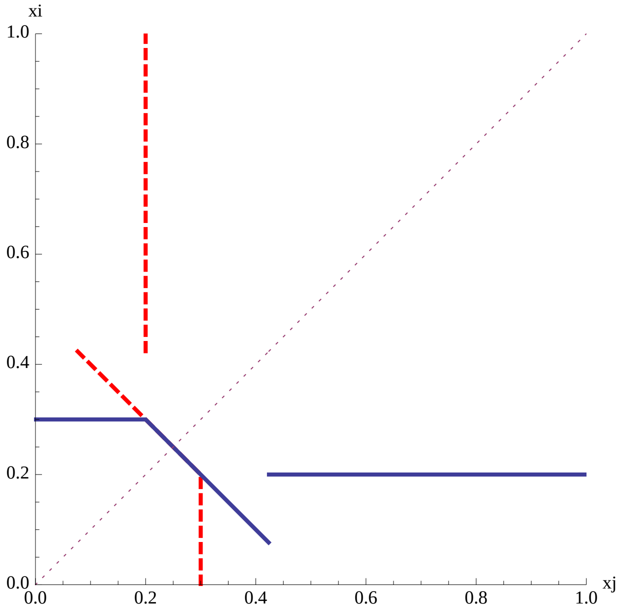

Figure 8: The solid line shows the best reply correspondence for investor i, whereas the dashed line depicts the same for investor j. The dotted line shows the identity function.

Consider the symmetric Nash equilibria in pure strategies. Analogously to Lemma 1, the aggregate demand function $x(p)$ can be derived assuming symmetric strategies for i and j.

Lemma 3. *In a symmetric Nash equilibrium in pure strategies of the game between the two investors the aggregate demand for shares $x(p)$ is piecewise linear of the form*

$$x(p) = \begin{cases} 2\frac{\mu-p}{a\sigma^2} & \text{if } p \leq \underline{p}, \\ 1-\frac{c}{m} & \text{if } \underline{p} \leq p \leq \bar{p}, \\ 2\frac{\mu+m-p}{a\sigma^2} & \text{if } \bar{p} \leq p, \end{cases}$$

where $\underline{p} = \mu + \frac{m}{4} - \frac{a\sigma^2(1-c/m)}{2} - \frac{\sqrt{m^2/4 + a\sigma^2(m-c)}}{2}$ *and* $\bar{p} = \mu + m - \frac{a\sigma^2(1-c/m)}{2}$, *with a discontinuity at* $p = \underline{p}$.

Proof. Start with the third part of the best reply correspondence in (4). It says that the optimal demand of investor i is

$$x_i = \frac{\mu+m-p}{a\sigma^2} \quad \text{if} \quad 1 - c/m - x_j \geq \frac{\mu+m-p}{a\sigma^2}.$$

Assuming symmetric strategies allows to substitute $x_j = \frac{\mu+m-p}{a\sigma^2}$ to the inequality, which yields

$$\begin{aligned}\frac{\mu+m-p}{a\sigma^2} &\le 1-c/m-\frac{\mu+m-p}{a\sigma^2}\\ \mu+m-p &\le \frac{a\sigma^2}{2}(1-c/m)\\ p &\ge \mu+m-\frac{a\sigma^2}{2}(1-c/m)\equiv \bar{p}.\end{aligned}$$

Now take the first part of the best reply correspondence in (4). It says that the optimal demand of investor i is

$$x_i=\frac{\mu-p}{a\sigma^2}\quad\text{if}\quad(\mu+m-p)(1-c/m-x_j)-\frac{a\sigma^2}{2}(1-c/m-x_j)^2\le\frac{(\mu-p)^2}{2a\sigma^2}.$$

Assuming symmetric strategies allows substitute $x_j=\frac{\mu-p}{a\sigma^2}$ to the inequality:

$$\begin{aligned}(\mu+m-p)(1-c/m-\frac{\mu-p}{a\sigma^2})-\frac{a\sigma^2}{2}(1-c/m-\frac{\mu-p}{a\sigma^2})^2 &\le \frac{(\mu-p)^2}{2a\sigma^2}\\ (\mu+m-p)(1-c/m)-m\frac{\mu-p}{a\sigma^2}-\frac{a\sigma^2(1-c/m)^2}{2}+(1-c/m)(\mu-p) &\le \frac{2(\mu-p)^2}{a\sigma^2}.\end{aligned}$$

If this inequality holds as an equality, it defines a quadratic equation in p. Similarly to the previous model, the smaller root of this equation provides the relevant upper bound on p,

$$p\le\frac{m/2+2\mu-a\sigma^2(1-c/m)-\sqrt{m^2/4+a\sigma^2(m-c)}}{2}\equiv\underline{p}.$$

□

To find the optimal price from the owner's perspective, proceed in two steps, as in the previous model. First, for each part of the demand function in Lemma 3, derive the locally optimal price for the owner. After that, calculate indirect utilities for the three local solutions and determine what the global solution is, depending on the exogenous parameters of the model.

Lemma 4. *The optimal price p^* is derived for each segment of the investor's demand function given in Lemma 3.*

1. *In the first segment, where $x(p)=2\frac{\mu-p}{a\sigma^2}$, the optimal price is*

$$p^*=\min\{\mu-\frac{a\sigma^2}{4},\underline{p}\},\quad\textit{where}\quad c\ge\frac{a\sigma^2m-2m^2+\sqrt{2a\sigma^2m^3+4m^4}}{2a\sigma^2}\equiv\bar{c}$$

   has to hold for an interior optimum.

2. *In the second segment, where demand is constant at $x(p)=1-c/m$, the optimal price is*

$$p^*=\bar{p}=\mu+m-\frac{a\sigma^2(1-c/m)}{2}.$$

3. In the third segment, where $x(p) = 2\frac{\mu+m-p}{a\sigma^2}$*, the optimal price is*

$$p^* = \max\{\mu + m - \frac{a\sigma^2}{4}, \bar{p}\},$$

where $c \leq \frac{m}{2}$ *has to hold for an interior optimum.*

Proof. For the first part of the Lemma, substitute $x = 2\frac{\mu-p}{a\sigma^2}$ and $y = 0$ to the utility function of the owner in (1). Maximizing with respect to p yields $p^* = \mu - \frac{a\sigma^2}{4}$. Then check under which conditions the constraint $p^* \leq \underline{p}$ holds, i.e.

$$\mu - \frac{a\sigma^2}{4} \leq \mu + m/4 - 0.5a\sigma^2(1 - c/m) - 0.5\sqrt{m^2/4 + a\sigma^2(m - c)}.$$

This can be transformed to the following quadratic inequality in c,

$$\frac{m^2}{4} + a\sigma^2(m - c) \leq \frac{(m + a\sigma^2)^2}{4} - a\sigma^2(1 - c/m)(m + a\sigma^2) + [a\sigma^2(1 - c/m)]^2.$$

When transforming this inequality one can see that a necessary condition for it to hold is

$$c \geq \frac{m}{2} - \frac{m^2}{2a\sigma^2}.$$

It turns out that of the two roots of the quadratic inequality in c, only the larger one satisfies the necessary condition. This larger root is the relevant lower bound for c in order for the constraint $p^* \leq \underline{p}$ to hold,

$$c \geq \frac{a\sigma^2 m - 2m^2 + \sqrt{2a\sigma^2 m^3 + 4m^4}}{2a\sigma^2}.$$

For the second part of the Lemma, note that the owner's utility is increasing in p. The only constraint is $\underline{p} \leq p \leq \bar{p}$, so the owner optimally chooses $p^* = \bar{p}$.

For the third part, substitute $x = 2\frac{\mu+m-p}{a\sigma^2}$ and $y = 1$ to (1). Maximizing with respect to p yields $p^* = \mu + m - \frac{a\sigma^2}{4}$. Then check under which conditions the constraint $p^* \geq \bar{p}$ holds,

$$\mu + m - \frac{a\sigma^2}{4} \geq \mu + m - \frac{a\sigma^2}{2}(1 - c/m) \quad \Leftrightarrow \quad c \leq \frac{m}{2}.$$

□

Analogously to the previous model, the final step in solving the extended model is to calculate the indirect utility of the owner for each price in Lemma 4 and find out what the globally optimal price for the owner is. In other words, the objective is to find the subgame-perfect Nash equilibria of the game. It turns out that depending on the exogenous parameters c, m, a and σ, there are three different types of equilibria. The following Proposition summarizes the results.

Proposition 2. *Depending on parameters, the following are subgame-perfect Nash equilibria.*

1. *If* $c^* \leq c < m$, *with* $c^* = \frac{a\sigma^2 m - m^2 + m\sqrt{2a\sigma^2 m + m^2}}{2a\sigma^2}$, *then* $p = \mu - \frac{a\sigma^2}{4}$, $x = \frac{1}{2}$, *and the owner does not monitor, i.e.* $y = 0$.
2. *If* $\frac{m}{2} \leq c \leq c^*$, *then* $p = \bar{p} = \mu + m - \frac{a\sigma^2}{2}(1 - c/m)$, $x = 1 - c/m$, *and the owner monitors, i.e.* $y = 1$.
3. *If* $0 < c \leq \frac{m}{2}$, *then* $p = \mu + m - \frac{a\sigma^2}{4}$, $x = \frac{1}{2}$, *and the owner monitors, i.e.* $y = 1$.

Proof. The structure of the proof is similar to Proposition 1. Starting from Lemma 4, calculate the owner's indirect utility for each of the three segments of the demand function. Then, compare the indirect utilities to find out which segment of the demand function the owner optimally chooses.

First, start with the interior solution in the first part of Lemma 4. Substituting $p = \mu - \frac{a\sigma^2}{4}$, $x = 2\frac{\mu - p}{a\sigma^2} = \frac{1}{2}$, and $y = 0$ to (1) yields an indirect utility of

$$U_O(\mu - \frac{a\sigma^2}{4}, \frac{1}{2}, 0) = \mu - \frac{a\sigma^2}{4}.$$

For the corner solution substitute $p = \underline{p}$, $x = 2\frac{\mu - \underline{p}}{a\sigma^2}$, and $y = 0$ to (1). After some calculations we obtain

$$U_O(\underline{p}, 2\frac{\mu - \underline{p}}{a\sigma^2}, 0) = \mu + (\frac{2c}{m} + \frac{m}{a\sigma^2} - 1)\sqrt{\frac{m^2}{4} + a\sigma^2(m - c)} - \frac{m^2}{2a\sigma^2} - \frac{m}{2} - \frac{a\sigma^2}{2} + \frac{a\sigma^2 c}{m} - \frac{a\sigma^2 c^2}{m^2}.$$

For the second part of Lemma 4, substitute $p = \bar{p}$, $x = 1 - c/m$, and $y = 1$ to (1), which yields

$$U_O(\bar{p}, 1 - c/m, 1) = \mu + m - c - \frac{a\sigma^2}{2}(1 - c/m) + \frac{a\sigma^2 c}{2m}(1 - c/m) - \frac{a\sigma^2 c^2}{2m^2}.$$

Finally, the same procedure applied to the interior solution of part three of Lemma 4 results in

$$U_O(\mu + m - \frac{a\sigma^2}{4}, \frac{1}{2}, 1) = \mu + m - c - \frac{a\sigma^2}{4}.$$

It is not necessary to calculate the indirect utility for the corner solution of part three, because it is identical to the indirect utility of part two.

Compare the interior solution of case 1 to the solution of case 2. After solving a quadratic inequality one can see that the interior solution of case 1 is preferred by the owner if

$$c \geq \frac{a\sigma^2 m - m^2 + m\sqrt{2a\sigma^2 m + m^2}}{2a\sigma^2} \equiv c^*.$$

We know from Lemma 4 that $c \leq \bar{c}$ has to hold for an interior solution in case 1. Comparing $\bar{c}$ and c^* yields

$$\begin{gathered}
\bar{c} \leq c^* \\
\frac{a\sigma^2 m - 2m^2 + \sqrt{2a\sigma^2 m^3 + 4m^4}}{2a\sigma^2} \leq \frac{a\sigma^2 m - m^2 + m\sqrt{2a\sigma^2 m + m^2}}{2a\sigma^2} \\
m \leq \sqrt{2a\sigma^2 m + m^2}.
\end{gathered}$$

It follows that the owner prefers the interior solution of case 1 to case two if $c \geq c^*$. It is unnecessary to compare the corner solution of case 1 to case 2, because the utility at the corner solution can never be higher than the utility at the interior solution.

Now compare the interior solution of case 3 to case 2. The owner prefers case 3 if

$$\mu + m - c - \frac{a\sigma^2}{4} \geq \mu + m - c - \frac{a\sigma^2}{2}(1 - c/m) + \frac{a\sigma^2 c}{2m}(1 - c/m) - \frac{a\sigma^2 c^2}{2m^2}.$$

After solving the quadratic inequality in c we see that the owner always prefers the interior solution of case 3 to case 2. It is not necessary to compare the corner solution of case 3 to case 2, because the two indirect utilities are identical.

Finally, compare the interior solutions of case 1 and case 3. From Lemma 4 we know that the feasibility constraint for case 1 is $c \geq \bar{c}$, whereas the feasibility constraint for case 3 is $c \leq \frac{m}{2}$. A simple calculation shows that the two regions never intersect. □

To visualize the results, fix the parameters m, a and σ and look at different values of c relative to m. Figure 9 shows how varying c is related to the three different types of subgame-perfect Nash equilibria in Proposition 2. The results are similar to the model with one outside investor. If, everything else equal, the costs of monitoring are low relative to the benefits of monitoring, then the owner sells $\frac{1}{2}$ of the firm at a high price and chooses to monitor. On the other hand, if the costs of monitoring are high relative to m, then he sells $\frac{1}{2}$ of the firm at a low price and does not monitor. For intermediate values of c there is another equilibrium where the investors acquire just enough shares to make the owner indifferent between monitoring and not monitoring, $x = 1 - c/m$. In equilibrium, however, the owner chooses to monitor.

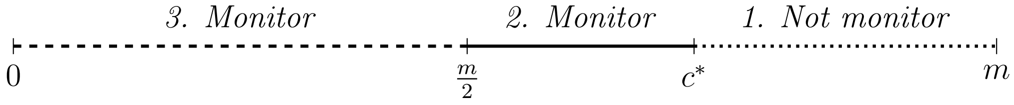

Figure 9: Depending of the values of c (for fixed m, a and σ), three different types of subgame-perfect Nash equilibria exist. The parameter that varies in this graph is c. The numbers 1, 2, and 3 refer to the numbering in Proposition 2. The constant c^* is also defined in Proposition 2.

Figure 10 shows the fraction of the firm x sold to the two investors, compared to the first-best sharing rule that maximizes social welfare. The picture reveals that the owner always sells less than the socially optimal amount. In other terms, risk sharing is never optimal. Moreover, for intermediate values of c the risk sharing between the firm owner and the two investors is distorted even further, as it was in the previous model. This corresponds to the equilibrium in part 2 of Proposition 2. The reason why risk sharing is worse in that equilibrium is that the investors acquire just enough shares to keep the owner indifferent between monitoring and not monitoring.

Similarly to the previous model, one can perform comparative statics with respect to risk. Again the parameters a and σ always show up jointly, so it makes sense to group them

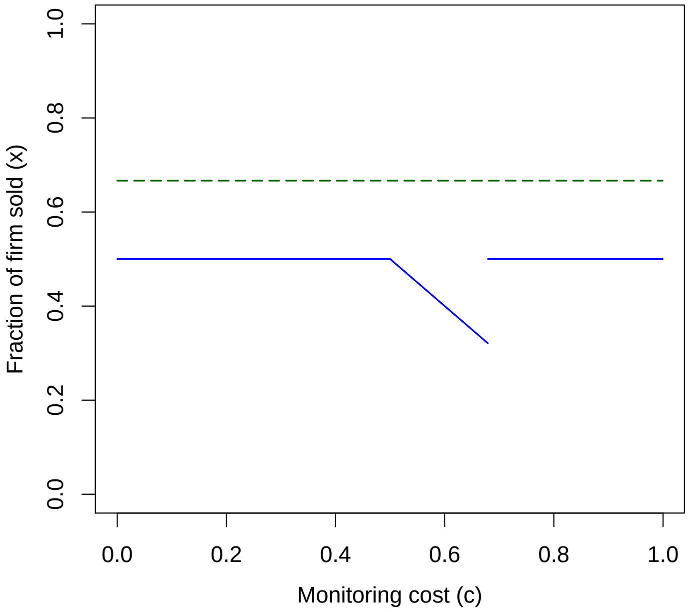

Figure 10: The solid line shows the fraction of the firm sold to the two investors, x, for different values of c. The dashed line depicts the optimal sharing rule from the perspective of a social planner. The graph is based on the parameter values $\sigma^2 = m = W = 1$, $\mu = 10$, $a = 10$.

together. The question is how the region of no monitoring, $c^* \leq c \leq m$, behaves if there are changes in $a\sigma^2$. A simple calculation shows that

$$\frac{\partial c^*}{\partial a\sigma^2} = \frac{\partial}{\partial a\sigma^2}\left[\frac{a\sigma^2 m - m^2 + m\sqrt{2a\sigma^2 m + m^2}}{2a\sigma^2}\right] < 0.$$

It follows that, ceteris paribus, the region with no monitoring is larger for high risk firms. The same is true if high risk is replaced with high risk aversion.

3.1 Welfare analysis

The welfare analysis of the model with two investors is very similar to the basic model. For the utility of the owner, substitute the results from Proposition 2 to (1). To simplify the analysis, and to make it comparable with the welfare analysis in the previous model, the welfare of the two outside investors is aggregated to

$$U_I(p, x, y) = \sum_{i=1,2} U_i(p, x, y) = W + x(\mu + ym - p) - \frac{a\sigma^2 x^2}{4}.$$

Then substitute the results from Proposition 2 to this expression. Figure 11 summarizes the welfare analysis in a stylized form. The dashed line in the middle shows the utility of the owner for varying values of c, the parameters a, σ, μ, W and m being constant.

The dotted line depicts the aggregate welfare of the two investors, and the solid line total welfare. The longdashed line on the top shows total welfare in a hypothetical economy where a social planner maximizes the sum of the three players' utilities. As in the model with a single investor, the players never reach the first-best total welfare. Also, total welfare is non-monotonic in c. The surprising result from the previous model that all players can be worse off by the monitoring technology no longer exists. It is an artifact of the simple model with only one investor. However, for intermediate values of c it still happens that total welfare is below the level without a monitoring technology. This happens at the discontinuity of the solid line in Figure 11, when c is below c^*.

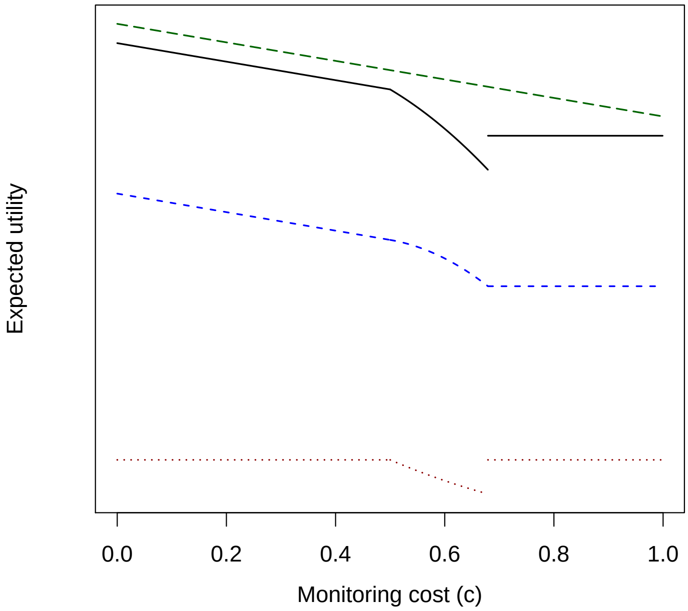

Figure 11: Utility of the owner (dashed line), total utility of the two investors (dotted line), and total welfare (solid line). The longdashed line on the top shows total welfare in a hypothetical economy where a social planner maximizes the sum of the three players' utilities. The graph is based on the parameter values $\sigma^2 = m = W = 1$, $\mu = 10$, $a = 10$.

4 Model with N investors

This is a straightforward extension of the model with two outside investors. There is a finite number N of investors, indexed by i. All investors are assumed to be symmetric in preferences and endowments and choose $x_i \geq 0$ to maximize the utility in (3). The initial wealth of each investor is replaced by $\frac{W}{N}$. The sum of their demands must not exceed total supply,

$$x \equiv \sum_{i=1}^{N} x_i \leq 1.$$

As in the model with two investors, the monitoring decision of the owner depends on x,

$$y = \begin{cases} 1 & \text{if } (1-x)m > c \\ 0 & \text{if } (1-x)m < c. \end{cases}$$

At $(1-x)m = c$ the owner is indifferent between monitoring and not monitoring.

Every price $p \geq 0$, chosen by the owner, initiates a subgame where the investors simultaneously decide how many shares to purchase. Analogously to the previous model, the best reply correspondence of investor i is

$$x_i = \begin{cases} \frac{\mu-p}{a\sigma^2} & \text{if } U_i(p, 1-c/m-x_{-i}, 1) \leq \max_{x_i} U_i(p, x_i, 0), \\ 1-c/m-x_{-i} & \text{if } U_i(p, 1-c/m-x_{-i}, 1) \geq \max_{x_i} U_i(p, x_i, 0) \\ & \text{and } 1-c/m-x_{-i} \leq \arg\max_{x_i} U_i(p, x_i, 1), \\ \frac{\mu+m-p}{a\sigma^2} & \text{else,} \end{cases}$$

where $x_{-i} = \sum_{j \neq i} x_j$. Writing this best reply correspondence explicitly yields

$$x_i = \begin{cases} \frac{\mu-p}{a\sigma^2} & \text{if } (\mu+m-p)(1-c/m-x_{-i}) - \frac{a\sigma^2}{2}(1-c/m-x_{-i})^2 \leq \frac{(\mu-p)^2}{2a\sigma^2}, \\ 1-c/m-x_{-i} & \text{if } (\mu+m-p)(1-c/m-x_{-i}) - \frac{a\sigma^2}{2}(1-c/m-x_{-i})^2 \geq \frac{(\mu-p)^2}{2a\sigma^2} \\ & \text{and } 1-c/m-x_{-i} \leq \frac{\mu+m-p}{a\sigma^2}, \\ \frac{\mu+m-p}{a\sigma^2} & \text{else.} \end{cases}$$

Assuming symmetric strategies for the investors allows to use the best reply correspondences to derive the aggregate demand function, that is $x(p)$.

Lemma 5. *In a symmetric Nash equilibrium in pure strategies of the game between the N investors there are thresholds $0 < \underline{p} < \mu$ and $\underline{p} < \bar{p} < \mu + m$ such that the aggregate demand for shares x is piecewise linear of the form*

$$x(p) = \begin{cases} N\frac{\mu-p}{a\sigma^2} & \text{if } p \leq \underline{p}, \\ 1-\frac{c}{m} & \text{if } \underline{p} \leq p \leq \bar{p}, \\ N\frac{\mu+m-p}{a\sigma^2} & \text{if } \bar{p} \leq p. \end{cases}$$

The thresholds $\underline{p}$ and $\bar{p} = \mu + m - \frac{a\sigma^2(1-c/m)}{N}$ are increasing in N, with $\lim_{N\to\infty} \underline{p} = \mu$ and $\lim_{N\to\infty} \bar{p} = \mu + m$.

Proof. Start with part 1 of the best reply correspondence of investor i, and insert the demand function $x_i = \frac{\mu-p}{a\sigma^2}$ for all players. This provides a quadratic inequality in p, and we know from the previous two models that the smaller root provides an upper bound for the price $p \leq \underline{p}$. It is not so easy anymore to obtain a closed form expression for $\underline{p}$. However, the Implicit Function Theorem can be used to analyze how $\underline{p}$ changes with N. For this define $n = N - 1$ and the function

$$G(p, n) = (\mu+m-p)(1-c/m-n\frac{\mu-p}{a\sigma^2}) - \frac{a\sigma^2}{2}(1-c/m-n\frac{\mu-p}{a\sigma^2})^2 - \frac{(\mu-p)^2}{2a\sigma^2}.$$

The derivative $\frac{\partial G}{\partial p}$ is positive,

$$\frac{\partial G}{\partial p} = (n+1)\left[\frac{\mu-p}{a\sigma^2} - (1-c/m-n\frac{\mu-p}{a\sigma^2})\right] + \frac{nm}{a\sigma^2} > 0,$$

because the term in square brackets is positive, and so are the other terms. The derivative $\frac{\partial G}{\partial n}$ is negative,

$$\frac{\partial G}{\partial n} = -(\mu - p)\left[\frac{\mu + m - p}{a\sigma^2} - (1 - c/m - n\frac{\mu - p}{a\sigma^2})\right] < 0,$$

because the term is square brackets is positive and the term in parentheses in positive as well. It follows from the Implicit Function Theorem that $d\underline{p}/dn > 0$.

It turns out that $\underline{p}$ does not grow without bound as $n = N - 1$ increases. Take the function $G(p, n)$, and take the limit

$$\lim_{n\to\infty} G(p, n) = -\frac{\mu - p}{a\sigma^2}.$$

The reason why this limit is so simple is that the term $(1 - c/m - n\frac{\mu - p}{a\sigma^2})$ converges to zero as n grows. This is because the term in parentheses is the amount of stock investor i has to hold if all investors collectively want to make the owner indifferent between monitoring and not monitoring. By assumption all investors use symmetric strategies, hence the share of investor i converges to zero. This however provides a value for $\underline{p}$ in the limit,

$$\lim_{n\to\infty} \underline{p} = \mu.$$

In the second part of the best reply correspondence of investor i it is clear that if each investor holds $x_i = 1 - c/m - x_{-i}$, then aggregate demand is $x = 1 - c/m$.

For the third part of the proof, assuming $x_i = \frac{\mu + m - p}{a\sigma^2}$ for every investor yields the inequality

$$\begin{aligned}\frac{\mu + m - p}{a\sigma^2} &\leq 1 - c/m - x_{-i}\\ \frac{\mu + m - p}{a\sigma^2} &\leq 1 - c/m - n\frac{\mu + m - p}{a\sigma^2}\\ p &\geq \mu + m - \frac{a\sigma^2(1 - c/m)}{n + 1} \equiv \bar{p}.\end{aligned}$$

It is easy to see that $\bar{p}$ is increasing in $N = n + 1$ and that $\lim_{N\to\infty} \bar{p} = \mu + m$. □

Given the aggregate demand function for stocks the owner chooses a price p to maximize his expected utility. In a first step, the optimal price for each segment of the demand function is derived.

Lemma 6. *The optimal price p^* is derived for each segment of the investor's demand function given in Lemma 5.*

1. *In the first segment, where $x(p) = N\frac{\mu - p}{a\sigma^2}$, the optimal price is*

$$p^* = \min\{\mu - \frac{a\sigma^2}{2 + N}, \underline{p}\}.$$

2. *In the second segment, where demand is* $x(p) = 1 - c/m$*, the optimal price is*

$$p^* = \bar{p} = \mu + m - \frac{a\sigma^2(1 - c/m)}{N}.$$

3. *In the third segment, where* $x(p) = N\frac{\mu+m-p}{a\sigma^2}$*, the optimal price is*

$$p^* = \max\{\mu + m - \frac{a\sigma^2}{2+N}, \bar{p}\}.$$

Proof. For the first part, substitute $x(p) = N\frac{\mu-p}{a\sigma^2}$ and $y = 0$ to the owner's utility function in (1). Then maximize with respect to p, subject to $p \leq \underline{p}$. For the second part, insert $x(p) = 1 - c/m$ and $y = 1$ to (1) and maximize with respect to p, subject to $\underline{p} \leq p \leq \bar{p}$. Finally, repeat the procedure for $x(p) = N\frac{\mu+m-p}{a\sigma^2}$, $y = 1$, and the constraint $p \geq \bar{p}$. ☐

The final step to the solution of the model with N investors is to find the globally optimal price p for the owner, based on the local solutions in Lemma 6. Since $\underline{p}$ is not available in closed form, we have to rely on the interior optima of Lemma 6. A justification for this is that in the limit as N grows to infinity, the interior solution of part 1 is always feasible. This can be seen by substituting $p = \mu - \frac{a\sigma^2}{2+N}$ to the inequality in the first part of the best reply correspondence of investor i, together with $x_i = \frac{\mu-p}{a\sigma^2}$ for every player $i = 1, ..., N$. A simple calculation shows that the inequality always holds if N is sufficiently large. Also, it is sufficient to use the interior solution of part 3 of Lemma 6, because the corner solution is identical to the solution in part 2.

Proposition 3. *Assume the number of investors* N *grows without bound. Depending on parameters, the following are subgame-perfect Nash equilibria.*

1. *If* $c^* \leq c < m$*, with* $c^* = \frac{-m^2+m\sqrt{2a\sigma^2 m+m^2}}{a\sigma^2}$*, then* $p \to \mu$*,* $x \to 1$*, and the owner does not monitor, i.e.* $y = 0$*.*
2. *If* $0 < c \leq c^*$*, then* $p \to \mu + m$*,* $x = 1 - c/m$*, and the owner monitors, i.e.* $y = 1$*.*

Proof. Calculate the owner's indirect utility for each segment of the investors' demand function. For the first part, substitute $p = \mu - \frac{a\sigma^2}{2+N}$, $x(p) = N\frac{\mu-p}{a\sigma^2} = \frac{N}{2+N}$, and $y = 0$ to the owner's utility function in (1). This yields

$$U_O(\mu - \frac{a\sigma^2}{2+N}, \frac{N}{2+N}, 0) = \mu - \frac{a\sigma^2}{2+N}.$$

For the second part of the demand function, the same procedure results in

$$U_O(\mu + m - \frac{a\sigma^2(1 - c/m)}{N}, 1 - c/m, 1) = \mu + m - c - \frac{a\sigma^2}{N} + \frac{2a\sigma^2 c}{Nm} - c^2\left(\frac{a\sigma^2}{Nm^2} + \frac{a\sigma^2}{2m^2}\right).$$

The third segment of the demand function corresponds to an indirect utility of

$$U_O(\mu + m - \frac{a\sigma^2}{2+N}, \frac{N}{2+N}, 1) = \mu + m - c - \frac{a\sigma^2}{2+N}.$$

The price that corresponds to the interior optimum of case 3 is feasible if

$$\begin{aligned} p &\geq \bar{p} \\ \mu + m - \frac{a\sigma^2}{2+N} &\geq \mu + m - \frac{a\sigma^2(1-c/m)}{2+N} \\ m\left(1 - \frac{N}{2+N}\right) &\geq c. \end{aligned}$$

As N grows without bound this converges to $c \leq 0$. Since by assumption $c > 0$, in the limit the equilibrium in case 3 is never feasible.

The owner prefers case 1 to case 2 if

$$\mu - \frac{a\sigma^2}{2+N} \geq \mu + m - c - \frac{a\sigma^2}{N} + \frac{2a\sigma^2 c}{Nm} - c^2\left(\frac{a\sigma^2}{Nm^2} + \frac{a\sigma^2}{2m^2}\right).$$

If N grows without bound then this inequality simplifies to

$$m - c - \frac{c^2 a\sigma^2}{2m^2} \geq 0,$$

which yields that case 1 is preferred if

$$c \geq \frac{-m^2 + m\sqrt{2a\sigma^2 m + m^2}}{a\sigma^2} \equiv c^*.$$

□

To visualize the results in Proposition 3, fix the parameters m, a and σ, and vary the cost of monitoring c in the range $(0, m)$. Figure 12 shows that for small values of c relative to m, the investors buy just enough shares to keep the owner indifferent between monitoring and not monitoring. In equilibrium he monitors, which is reflected in the high stock price $p \to \mu + m$. However, for large values of c relative to m, in the limit the owner sells the whole firm to the investors at a low price, $p \to \mu$, and does not monitor.

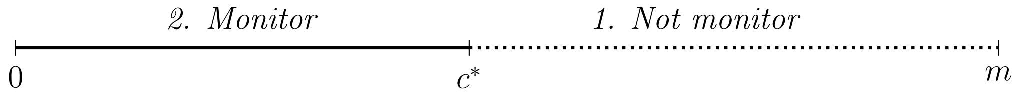

Figure 12: Depending of the values of c (for fixed m, a and σ), two types of subgame-perfect Nash equilibria exist. The parameter that varies in this graph is c. The numbers 1 and 2 refer to the numbering in Proposition 3. The constant c^* is also defined in Proposition 3.

The threshold $c^* = \frac{-m^2+m\sqrt{2a\sigma^2 m+m^2}}{a\sigma^2}$ depends on m and the product $a\sigma^2$. A simple calculation shows that

$$\frac{\partial c^*}{\partial a\sigma^2} < 0.$$

This means that for firms with higher risk, the region with no monitoring expands. The higher risk can equivalently be interpreted as a higher risk aversion. However, since it is assumed that all players have the same risk aversion, the first interpretation is more useful.

It is also interesting to see what fraction of the firm is sold to investors for different values of c. Figure 13 shows that for $c \geq c^*$, in the limiting case of $N \to \infty$, the owner sells the whole firm to investors. However, for $c \leq c^*$ the fraction of the firm sold is decreasing in c. This is the region where investors keep the owner indifferent between monitoring and not monitoring. With an infinite number of investors, the first-best equilibrium would be to sell the whole firm for any value of c, at a price $p = \mu + m$, which would correspond in perfect risk sharing. However, in the region $c \leq c^*$ risk sharing is distorted due to the moral hazard problem between the investors and the owner. Figure 13 also summarizes the trade-off between monitoring and risk sharing. For risky firms the threshold c^* is small. Therefore, for most values of c, the owner sells the whole firm, which is equivalent to full risk sharing. On the other hand, for firms with lower risk the threshold c^* is large. This implies that the discontinuity in Figure 13 is closer to the right end of the graph. For most values of c, the investors and the owner are far away from perfect risk sharing. This is intuitive beause for firm with less risk, agents care less about risk sharing and more about the benefits from monitoring.

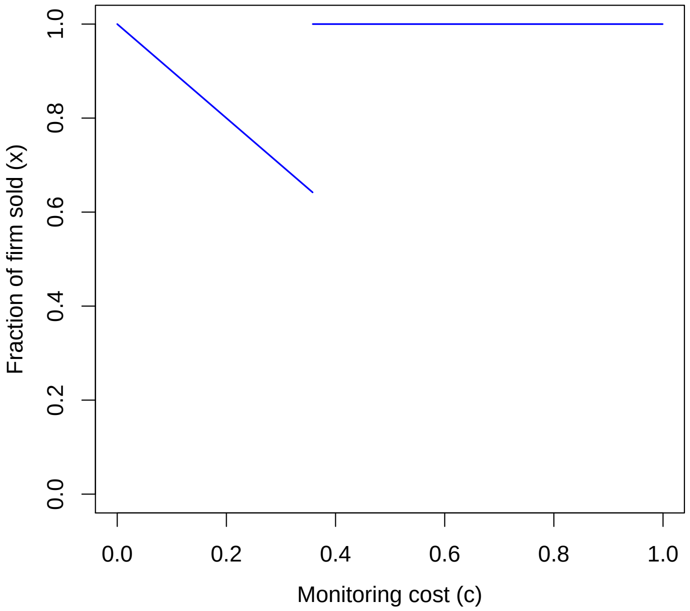

Figure 13: The line shows the fraction of the firm sold to the two investors, x, for different values of c. The graph is based on the parameter values $\sigma^2 = m = W = 1$, $\mu = 10$, $a = 10$.

4.1 Welfare analysis

For the owner's utility substitute the results from Proposition 3 to his utility function in (1). The sum of all investors' utilities is given by

$$U_I(p, x, y) = \sum_{i=1}^{N} U_i(p, x, y) = W + x(\mu + ym - p) - \frac{a\sigma^2 x^2}{2N}.$$

In the limit as $N \to \infty$, this converges to $U_I = W$. Figure 14 summarizes the welfare analysis in one picture. The dashed line in the middle shows the utility of the owner for varying values of c, the parameters a, σ, μ, W and m being constant. The dotted line depicts the aggregate welfare of the investors, and the solid line total welfare. The longdashed line on the top shows total welfare in a hypothetical economy where a social planner maximizes the sum of the three players' utilities. As in the previous two models, total welfare never reaches the socially optimal level. However, if c is close to either 0 or m, total welfare converges to the first-best level. In contrast to the previous models, total utility is now a monotonically decreasing function of c. Also, total welfare is never below the level it would be if there was no monitoring technology. The welfare of the investors is constant at $U_I = W$, which means that the surplus generated by the monitoring technology is fully captured by the owner. An increase in firm risk, σ, affects the welfare of the owner because the threshold c^* decreases. In Figure 14 this means that the kink of the dashed line in the middle and of the solid line move to the left. The utility of the owner is reduced by such increases is risk, whereas the investors are unaffected.

5 Conclusion

This paper presents a model of a firm owner's trade-off between monitoring and risk sharing. Some of the results of the paper are very intuitive: In a subgame-perfect Nash equilibrium there are situations where monitoring does not take place, although it would be socially desirable. The two players do not reach the same outcome that a social planner would impose. Also, there is less risk sharing than in the social optimum. This result is consistent with previous papers on this issue.

On the other hand, the solution of the model also reveals some surprising results. Foremost in some cases it is possible that the availability of the monitoring technology makes *both* players worse off than they would be without a monitoring technology. The reason is that in such an equilibrium risk sharing is strongly distorted downwards. This is because the investor chooses his optimal demand for shares in a way to incentivize the owner to monitor. The investor could demand more shares and thereby improve risk sharing, but then he would destroy the owner's incentive to monitor. This extra distortion in risk sharing is caused by the assumption that the outside investor is strategic.

Another advantage of the model is that it's formal analysis can be interpreted in several ways. The interpretation chosen here is that a firm owner can monitor the activities within the firm, and thereby increase firm value. Another interpretation is that an entrepreneur can exert effort, which then increases the value of the firm. Both interpretations are consistent

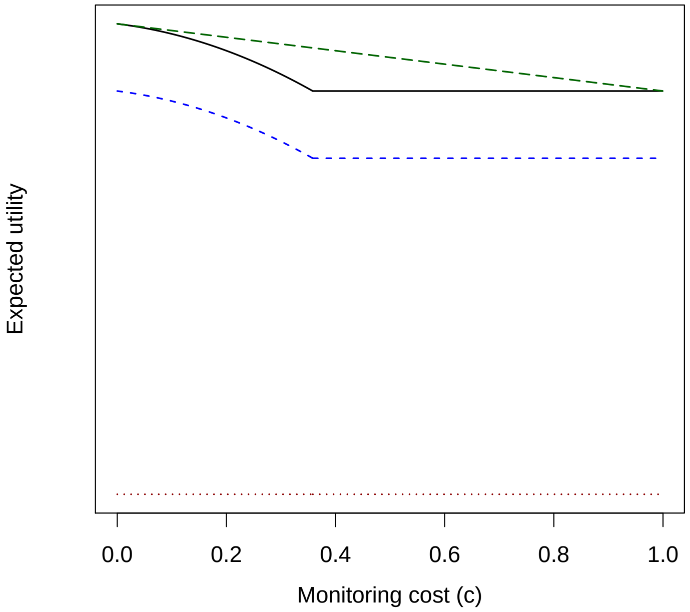

Figure 14: Utility of the owner (dashed line), total utility of the investors (dotted line), and total welfare (solid line). The longdashed line on the top shows total welfare in a hypothetical economy where a social planner maximizes the sum of all players' utilities. Utilities correspond to the limiting case $N \to \infty$. The graph is based on the parameter values $\sigma^2 = m = W = 1$, $\mu = 10$, $a = 10$.

with modeling the firm as a random variable where the mean of the random variable gets shifted.

The model extends the existing literature on monitoring by large shareholders by analyzing the problem from a game theoretic perspective. If both the owner and the investor are strategic, then extra distortions in risk sharing occur. The results of the model show that the strategic perspective can yield interesting insights that are not captured by traditional models.

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Appendix

Symbol	Meaning
O	Owner of the firm (first player)
I	Outside investor (second player)
D	Normally distributed firm payoff
D'	Normally distributed firm payoff if O monitors
μ	Mean of D
σ	Standard deviation of D
m	$E(D') - E(D)$
c	Cost of monitoring
W	Initial wealth of I
p	Stock price as chosen by O
x	Number of stocks acquired by I
r	Riskfree interest rate
W_T	Terminal wealth
u	Bernoulli utility function, CARA
U	von Neumann-Morgenstern utility function, $U \equiv E(u)$
a	Coefficient of absolute risk aversion, for both players
y	$y = 1$ if O monitors, $y = 0$ if he does not monitor

Table 1: Notation used in the model.