Asymmetric Volatility Smile in a Continuous-Time Agency Model. A Note on DeMarzo-Sannikov.

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Abstract

This note builds on a model by DeMarzo, Sannikov (2006), in order to show that an optimal principal-agent contract may give rise to an asymmetric volatility smile. The contribution of this note is stated in section 3. Stock price volatility increases together with the firm's total debt. Hence, in this model, an asymmetric volatility smile ia a function of credit risk.

1 Introduction

In this note I would like to show that contractual problems that arise from incentive constraints, present in all principal-agent models, may also be responsible for the stochastic volatility of stock prices. Additionally, I find that this stochastic volatility exhibits an asymmetric smile, which is very often observed in market implied volatilities. In order to show these properties I build on a continuous-time agency model from DeMarzo-Sannikov (2006, hereafter: DS).

For the convenience of the reader, section 2 provides a short description of the DS model. I do not repeat the proofs of DS, except for the two that I will need. The result of this note, which extends DeMarzo-Sannikov, is in section 3. Section 4 concludes.

2 Short summary of DeMarzo-Sannikov model

2.1 Optimal contract

An agent who manages a project generating a stochastic stream of cash flows given by a standard Brownian motion:

$$dY_t = \mu dt + \sigma dZ_t$$

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where Z_t is a standard Wiener process. The agent can privately observe cash flows and reports $\left\{\widehat{Y}_t, t>0\right\}$ of the realized cash flows. However, he may divert cash flows or manipulate them with his savings. That is, if the agent has positive savings, he may fabricate cash flows that did not actually realize. The agent can receive at most a fraction λ from diverted cash flows. So $1-\lambda$ is the dead-weight cost of "money laundering". Based on the reports and the extracted cash flows, the principal transfers a payoff to the manager of I_t. The agent's payoff process is assumed to be non-decreasing and $\widehat{Y}_t$ - measurable. The agent's flow of income consists of what he diverts plus his remuneration. The contract termination time τ is specified by the investor. Before τ the agent chooses a non-decreasing consumption process $C=\{C_t, 0\leq t\leq\tau\}$. Upon termination of the contract, the agent receives his reservation utility R and the investors receive the liquidation value L. The agent is assumed to be risk-neutral with the subjective time preference rate γ. The agent's total payoff from the contract at time 0 is given by

$$W_0=E\left[\int_0^\tau e^{-\gamma t}dC_t+e^{-\gamma\tau}R\right].$$

Investors are also assumed to be risk-neutral. They have an unlimited capital and they discount received cash flows at the rate r, such that $r<\gamma$. The principal's total payoff at date 0 is equal to

$$b\left(W_0\right)=E\left[\int_0^\tau e^{-rt}\left(d\widehat{Y}_t-dI_t\right)dt+e^{-r\tau}L\right].$$

Before time zero, the principal specifies a contract (τ, I) consisting of termination time τ and payments to the agent $\{I_t, 0\leq t\leq\tau\}$ based on the reports $\widehat{Y}_t$. The agent chooses his strategy $\left(C,\widehat{Y}\right)$ as a response to the contract (τ, I). The following proposition describes the optimal contract.

Proposition 1 *A contract that maximizes the principal's profit and delivers to the agent value $W_0\in\left[R, W^1\right]$ takes the following form: W_t evolves as*

$$dW_t=\gamma W dt-dI_t+\lambda\left(d\widehat{Y}_t-\mu dt\right)$$

When $W_t\in\left[R, W^1\right]$, $dI_t=0$. When $W_t=W^1$, payments dI_t cause W_t to reflect at W^1. If $W_0>W^1$ an immediate payment W_0-W^1 is made. The contract is terminated at time τ when W_t hits R. The principal's expected payoff is given by $b\left(W_t\right)$ which at any point on the interval $\left[R, W^1\right]$ satisfies

$$rb\left(W\right)=\mu+\gamma W b'\left(W\right)+\frac{1}{2}\lambda^2\sigma^2 b''\left(W\right)$$

and $b'\left(W\right)=-1$ for $W>W^1$, boundary conditions are $b\left(R\right)=L$ and $rb\left(W^1\right)=\mu-\gamma W^1$.

Proof. Please see the proof of proposition 1 in DS. ■

2.2 Capital structure implementation

The optimal contract is implemented with three standard securities:

Equity. Equity holders receive dividend payments made by the firm. Dividends are paid from the firm's available cash or credit and are at the discretion of the agent.

Long-term debt. Long-term debt is a consol bond that pays continuous coupons at the rate x. The coupon rate is r and the face value of debt is $D = x/r$.

Credit line A revolving credit line provides the firm with available credit up to C^L. Balances on the credit line are charged a fixed interest rate r^c. If the balance on the credit line exceeds C^L, the firm defaults.

The next proposition shows how the optimal contract can be implemented with the securities.

Proposition 2 *Consider a capital structure in which the agent holds inside equity for fraction λ of the firm, the credit line has interest rate $r^c = \gamma$ and the debt satisfies*

$$rD = \mu - \gamma R/\lambda - \gamma C^L.$$

Then it is incentive compatible for the agent to refrain from stealing and to use the project cash flows to pay the debt coupons and credit line before issuing dividends. Once the credit line is fully repaid, all excess cash flows are issued as dividends. Under this capital structure, the agent's expected future payoff W_t is determined by the current draw M_t on the credit line:

$$W_t = R + \lambda\left(C^L - M_t\right).$$

The capital structure implements the optimal contract if, in addition, the credit line satisfies

$$C^L = \lambda^{-1}\left(W^1 - R\right).$$

Proof. Please see the proof of proposition 3 in DS. ■

2.3 Security Market Values

First, we observe that the market value of long-term debt, credit line and outside equity is conditional on the draw on the credit line. That happens because the larger the draw on the credit line, the higher the probability of default. The long-term debt holders in case of default would get

$L_D = \min(L, D)$ and so the market value of debt is

$$V_D(M) = E\left[\int_0^\tau e^{-rt} x dt + e^{-r\tau} L_D | M\right].$$

Outside equity holders get $(1-\lambda) dDiv_t$ out of total dividends. At termination they get their fraction of what remains after debt and credit line have been paid off,

$$L_E = (1-\lambda)\max\left(0, L - D - C^L\right).$$

Therefore, the value of equity per share of outside equity holders is equal to

$$V_E(M) = E\left[\int_0^\tau e^{-rt} dDiv_t + e^{-r\tau} L_E | M\right]$$

and the market value of credit line is equal to

$$V_C(M) = E\left[\int_0^\tau e^{-rt}(dY_t - xdt - dDiv_t) + e^{-r\tau} L_C | M\right],$$

where $L_C = \min\left(C^L, L - L^D\right)$.

Differential equations representing the prices One of the main advantages of the continuous time setting is that we can represent values of the above securities in the form of differential equations. The next lemma guarantees such a convenient representation that avoids the use of expected values.

Lemma 3 *Suppose W_t evolves as*

$$dW_t = \gamma W_t dt - dI_t + \lambda\sigma dZ_t$$

in the interval $\left[R, W^1\right]$ until time τ when W_t hits R and bankruptcy occurs. I_t is a non-decreasing process that reflects W_t at W^1. Let k be a real number and $g : \left[R, W^1\right] \to \Re$ be a bounded function. When boundary conditions $G(R) = L$ and $G'\left(W^1\right) = -k$ are satisfied, then the following two conditions are equivalent

1. *function G solves*
$$rG(W) = g(W) + \gamma W G'(W) + \frac{1}{2}\lambda^2\sigma^2 G''(W)$$

2. *and function G satisfies*
$$G(W_0) = E\left[\int_0^\tau e^{-rt} g(W_t)\, dt - k\int_0^\tau e^{-r\tau} dI_t + e^{-r\tau} L\right].$$

Proof.

- $\Rightarrow$

  Suppose first that G solves

$$rG(W) = g(W) + \gamma W G'(W) + \frac{1}{2}\lambda^2\sigma^2 G''(W)$$

and based on that we will show that it also satisfies

$$G(W_0) = E\left[\int_0^\tau e^{-rt} g(W_t)\,dt - k\int_0^\tau e^{-r\tau} dI_t + e^{-r\tau} L\right].$$

Define the process

$$H_t = \int_0^t e^{-rs} g(W_s)\,ds - k\int_0^t e^{-rs} dI_s + e^{-rt} G(W_t)$$

then if we take the Ito derivative of H_t we get

$$e^{rt} dH_t = \underbrace{\left[g(W_t) + \gamma W_t G'(W_t) + \frac{1}{2}G''(W_t)\lambda^2\sigma^2 dt - rG(W_t)\right]}_{=0} dt - (k + G'(W_t))\,dI_t + G'(W_t)\lambda\sigma dZ_t$$

We also know that either $W_t < W^1$ and then $dI_t = 0$ or for $W_t \geq W^1$ and then $G'(W_t) = -k$, therefore we have shown that H is a martingale. Because G is bounded, H is a martingale until time τ, so that

$$G(W_0) = H_0 = E[H_\tau] = E\left[\int_0^\tau e^{-rt} g(W_t)\,dt - k\int_0^\tau e^{-r\tau} dI_t + e^{-r\tau} L\right]$$

- $\Leftarrow$ This direction is missing in DS.

■

Remark The proof of this lemma is not complete in DS, because it is an "if and only if" condition and it should be shown also in the other direction. However, to complete the proof is not very difficult and I will prove it both ways in Lemma 5, which is a slightly more general version of Lemma 3.

Now, in order to compute the values of credit line, debt and equity, the following two functions will be used,

$$G_\tau(W) = E\left[e^{-rt} | W_0 = W\right] \text{ and } G_I(W) = E\left[\int_0^\tau e^{-rt} dI_t | W_0 = W\right].$$

From Lemma 3, we know that both of these functions solve the differential equation:

$$rG(W) = \gamma W G'(W) + \frac{1}{2}\lambda^2\sigma^2 G''(W)$$

with boundary conditions $G_\tau(R) = 1$, $G'_\tau(W^1) = 0$ and $G_I(R) = 0$, $G'_I(W^1) = 1$. Using G_τ we can also compute

$$E\left[\int_0^\tau e^{-rt}|W_0 = W\right] = \frac{1}{r}(1 - G_\tau(W)).$$

Finally, formulas for the value of long-term debt, equity and credit line can be expressed as functions of the continuation value W_t,

$$\begin{aligned} V_D(M) &= \frac{x}{r}(1 - G_\tau(W)) + L_D G_\tau(W) \\ V_E(M) &= \frac{1}{\lambda}G_I(W) + L_E G_\tau(W) \\ V_C(M) &= \frac{1}{\lambda}\gamma W^1 \frac{1}{r}(1 - G_\tau(W)) - \frac{1}{\lambda}G_I(W) + L_C G_\tau(W), \end{aligned}$$

where $W_t = W^1 - \lambda M_t$.

We need the next lemma in order to describe the volatility of stock prices.

Lemma 4 *Function $G_I(W)$ is concave.*

Proof. Suppose that G_I were not concave somewhere on $[R, W^1]$, and let $V = \inf\{G''_I(W) > 0\}$. Then $V > R$ and $G''_I(V) = 0$ by continuity of G''_I. But then

$$1/2\lambda^2\sigma^2 G'''_I(V) = (r - \gamma)G'_I(V) - \gamma V G''_I(V) = (r - \gamma)G'_I(V) < 0$$

so $G''_I(V + \varepsilon) < 0$ for all sufficiently small $\varepsilon > 0$, which is a contradiction. ■

3 Equity price volatility

We know from the previous section that at time 0, if debt is risky, the value of equity is equal to

$$V_E(M) = E\left[\int_0^\tau e^{-rt}\frac{1}{\lambda}dI_t|W = W_0\right] = \frac{1}{\lambda}G_I(W), \text{ where } W = W^1 - \lambda M.$$

But we might be interested also in the evolution of the stock price, that is, how it behaves at time $t > 0$. This is very easy, we just need to condition on $\mathcal{F}_t^Z$ and integrate from t to τ rather than from 0 to τ. So let us define two new functions:

$$\begin{aligned} S(t, W_t) &= E\left[\int_t^\tau e^{-r(s-t)} \frac{1}{\lambda} dI_s | \mathcal{F}_t^Z, W = W_t\right] \\ F(t, W_t) &= E\left[\int_t^\tau e^{-r(s-t)} dI_s | \mathcal{F}_t^Z, W = W_t\right]. \end{aligned}$$

Obviously we have

$$S(t, W_t) = \frac{1}{\lambda} F(t, W_t)$$

and for $t = 0$, we have that $F(0, W_0) = G_I(W_0)$. We can now use $S(t, W_t)$ in order to investigate properties of the stock price volatility in the DS model. But before that we need to describe function F. The next lemma is a direct generalization of Lemma 3, but here the proof is complete.

Lemma 5 *Suppose W_t evolves as*

$$dW_t = \gamma W_t dt - dI_t + \lambda \sigma dZ_t$$

in the interval $[R, W^1]$ until time τ when W_t hits R and bankruptcy occurs. I_t is a non-decreasing process that reflects W_t at W^1. Let $F = F(t, W_t)$ be a C^2 function with boundary conditions $F(\tau, R) = 0$ and $\frac{\partial}{\partial W} F(t, W^1) = 1$. Then the following two equations are equivalent:

1. *Function F solves*

$$rF(t, W_t) = F_t(t, W_t) + \gamma W \frac{\partial}{\partial W} F(t, W_t) + \frac{1}{2} \lambda^2 \sigma^2 \frac{\partial^2}{\partial W^2} F(t, W_t)$$

2. *and F satisfies*

$$F(t, W_t) = E\left[\int_t^\tau e^{-r(s-t)} dI_s | \mathcal{F}_t^Z, W = W_t\right].$$

Proof.

- $\Rightarrow$

  Suppose that F solves $rF(t, W_t) = F_t(t, W_t) + \gamma W \frac{\partial}{\partial W} F(t, W_t) + \frac{1}{2} \lambda^2 \sigma^2 \frac{\partial^2}{\partial W^2} F(t, W_t)$ and let us show that it satisfies $F(t, W_t) = E\left[\int_t^\tau e^{-r(s-t)} dI_s | \mathcal{F}_t^Z, W = W_t\right]$. Define

$$\widetilde{H}_u = \int_t^u e^{-r(s-t)} dI_s + e^{-r(u-t)} F(u, W_u).$$

Then, using Ito's lemma,

$$\begin{aligned} e^{r(u-t)}d\widetilde{H}_u &= dI_u - rF(u,W_u)\,du + F_u(u,W_u)\,du + \gamma W\frac{\partial}{\partial W}F(u,Wu)\,du - \frac{\partial}{\partial W}F(u,W_u)\,dI_u + \\ &+\lambda\sigma\frac{\partial}{\partial W}F(u,W_u)\,dZ_u + \frac{1}{2}\lambda^2\sigma^2\frac{\partial^2}{\partial W^2}F(u,W_u)\,du \end{aligned}$$

and after rearranging,

$$\begin{aligned} e^{r(u-t)}d\widetilde{H}_u &= \underbrace{\left\{F_u(u,W_u) + \gamma W\frac{\partial}{\partial W}F(u,Wu) + \frac{1}{2}\lambda^2\sigma^2\frac{\partial^2}{\partial W^2}F(u,W_u) - rF(u,W_u)\right\}}_{=0}du + \\ &+\left(1-\frac{\partial}{\partial W}F(u,W_u)\right)dI_u + \lambda\sigma\frac{\partial}{\partial W}F(u,W_u)\,dZ_u. \end{aligned}$$

From this equation we get

$$e^{r(u-t)}d\widetilde{H}_u = \left(1-\frac{\partial}{\partial W}F(u,W_u)\right)dI_u + \lambda\sigma\frac{\partial}{\partial W}F(u,W_u)\,dZ_u$$

and we observe that either $W_u < W^1$ and then $dI_u = 0$ or $W_u \geq W^1$ and then $\frac{\partial}{\partial W}F(u,W_u) = 1$, so $\widetilde{H}_u$ is a martingale, which implies:

$$F(t,W_t) = \widetilde{H}_t = E\left[\widetilde{H}_\tau\right] = E\left[\int_t^\tau e^{-r(s-t)}dI_s|\mathcal{F}_t^Z\right].$$

- $\Leftarrow$

It is sufficient to show that $\widetilde{H}_u$ is a martingale if $F(t,W_t)$ solves $E\left[\int_t^\tau e^{-r(s-t)}dI_s|\mathcal{F}_t^Z\right]$. So let $\widetilde{H}_u$ be

$$\widetilde{H}_u = \int_t^u e^{-r(s-t)}dI_s + e^{-r(u-t)}F(u,W_u).$$

We can show that $\widetilde{H}_u$ is a martingale using the law of iterated expectations. Let us assume that $t < u < t' < \tau$ and then we need to show that $E\left[E\left[\widetilde{H}|\mathcal{F}_{t'}\right]|\mathcal{F}_u\right] = E\left[\widetilde{H}|\mathcal{F}_u\right]$. First let us define

$$\begin{aligned}
\widetilde{H}_{t'} &= \int_t^{t'} e^{-rs} dI_s + e^{-rt'} F\left(t', W_{t'}\right) = E\left[\int_t^{t'} e^{-rs} dI_s + e^{-rt'} F\left(t', W_{t'}\right) |\mathcal{F}_{t'}^Z\right] = E\left[\widetilde{H}_{t'} |\mathcal{F}_{t'}^Z\right] \\
\widetilde{H}_u &= \int_t^u e^{-rs} dI_s + e^{-ru} F\left(u, W_u\right) = E\left[\int_t^u e^{-rs} dI_s + e^{-ru} F\left(u, W_u\right) |\mathcal{F}_u^Z\right] = E\left[\widetilde{H}_u |\mathcal{F}_u^Z\right] \\
F\left(t', W_{t'}\right) &= E\left[\int_{t'}^{\tau} e^{-r\left(s-t'\right)} dI_s |\mathcal{F}_{t'}^Z\right] \\
F\left(u, W_u\right) &= E\left[\int_u^{\tau} e^{-r(s-u)} dI_s |\mathcal{F}_u^Z\right].
\end{aligned}$$

With these we can show that the low of iterated expectations holds:

$$\begin{aligned}
& E\left[E\left[\widetilde{H}_{t'} |\mathcal{F}_{t'}^Z\right] |\mathcal{F}_u^Z\right] \\
= & E\left[\int_t^{t'} e^{-rs} dI_s + e^{-rt'} F\left(t', W_{t'}\right) |\mathcal{F}_u^Z\right] \\
= & E\left[\int_t^u e^{-rs} dI_s + \int_u^{t'} e^{-rs} dI_s + e^{-rt'} E\left[\int_{t'}^{\tau} e^{-r\left(s-t'\right)} dI_s |\mathcal{F}_{t'}^Z\right] |\mathcal{F}_u^Z\right] \\
= & E\left[\int_t^u e^{-rs} dI_s + e^{-rt'} E\left[\int_u^{t'} e^{-rs} e^{rt'} dI_s |\mathcal{F}_u^Z\right] + e^{-rt'} E\left[\int_{t'}^{\tau} e^{-r\left(s-t'\right)} dI_s |\mathcal{F}_{t'}^Z\right] |\mathcal{F}_u^Z\right] \\
= & E\left[\int_t^u e^{-rs} dI_s + e^{-rt'} \left\{E\left[\int_u^{t'} e^{-r\left(s-t'\right)} dI_s |\mathcal{F}_u^Z\right] + E\left[\int_{t'}^{\tau} e^{-r\left(s-t'\right)} dI_s |\mathcal{F}_{t'}^Z\right]\right\} |\mathcal{F}_u^Z\right] \\
= & E\left[\int_t^u e^{-rs} dI_s + e^{-rt'} \left\{E\left[\int_u^{t'} e^{-r\left(s-t'\right)} dI_s |\mathcal{F}_u^Z\right] + E\left[\int_{t'}^{\tau} e^{-r\left(s-t'\right)} dI_s |\mathcal{F}_u^Z\right]\right\} |\mathcal{F}_u^Z\right] \\
= & E\left[\int_t^u e^{-rs} dI_s + e^{-rt'} E\left[\int_u^{t'} e^{-r\left(s-t'\right)} dI_s + \int_{t'}^{\tau} e^{-r\left(s-t'\right)} dI_s |\mathcal{F}_u^Z\right] |\mathcal{F}_u^Z\right] \\
= & \int_t^u e^{-rs} dI_s + e^{-rt'} e^{-r\left(-t'+u\right)} E\left[\int_u^{\tau} e^{-r(s-u)} dI_s |\mathcal{F}_u^Z\right] \\
= & \int_t^u e^{-rs} dI_s + e^{-ru} F\left(u, W_u\right) = \widetilde{H}_u = E\left[\widetilde{H}_u |\mathcal{F}_u^Z\right]
\end{aligned}$$

which proves that $\widetilde{H}_u$ is a martingale, so its drift must be equal to zero.

■

The following proposition describes the main result of this note. It states that an optimal contract between the investors and the entrepreneur may induce stochastic volatility in the stock prices and an asymmetric volatility smile.

Proposition 6 *When debt is risky, the volatility σ_E of equity value $S(t, W_t)$ is stochastic for all $W_t < W^1$. Stock price volatility σ_E increases with the amount drawn on the credit line M_t. For all values $W \geq W^1$ the volatility of equity is flat. Therefore, stock price volatility σ_E in this model exhibits the property called asymmetric volatility smile.*

Proof. Since the stock price $S(t, W_t)$ differs from $F(t, W_t)$ only by a constant factor of $\frac{1}{\lambda}$, then we can use the above lemma 6, to find the Ito derivative of

$$dS(t, W_t) = \left\{ S_t(t, W_t) + \gamma W S_W(t, W_t) + \frac{1}{2}\lambda^2 \sigma S_{WW}(t, W_t) \right\} dt + (1 - S_W(t, W_t))\, dI_t + \underbrace{\sigma S_W(t, W_t)}_{=\sigma_E} dZ_t$$

The Ito dynamics of S gives us the stock price volatility:

$$\sigma_E(t, W) = \sigma S_W(t, W_t)$$

or equivalently

$$\sigma_E(t, W) = \frac{1}{\lambda} \sigma F_W(t, W_t)$$

Now, we need to describe some of the properties of function F. First, function F is increasing and concave in W, that is for every t

$$\begin{aligned} F_W(t, W_t) &> 0 \\ F_{WW}(t, W_t) &< 0. \end{aligned}$$

In order to show that we need to observe that for fixed t

$$\frac{\partial}{\partial W} \left\{ \frac{\partial}{\partial t} F(t, W) \right\} = 0$$

and the rest of the proof follows exactly the same steps as the proof of Lemma 4.

Additionally, from $W_t = W^1 - \lambda M_t$ we get

$$\frac{\partial}{\partial M} \sigma_E(t, W) = \frac{\partial}{\partial M} \left(\frac{1}{\lambda} \sigma F_W(t, W_t) \right) = \frac{1}{\lambda} \sigma \frac{\partial}{\partial W} F_W(t, W) \times \underbrace{\frac{\partial W}{\partial M}}_{=-\lambda} = -\sigma \underbrace{F_{WW}(t, W)}_{<0} > 0.$$

Summarizing, we can see that the volatility of the stock price is increasing in M:

$$\frac{\partial}{\partial M} \sigma_E > 0,$$

that is increasing in the draw on the credit line. The agent has to draw on the credit line when the cash flows are not sufficient to repay the coupon on the long-term debt. This also increases a firm's leverage and credit risk. Altogether it means that when cash flows drop, stock price volatility increases. So we get an asymmetric stochastic volatility smile with a grin up in the direction of negative cash flows. ■

4 Conclusions

The interesting conclusion of this note is that the model is able to pin down a pattern of stock price behavior that is usually attributed just to the human component. Under standard Brownian motion, prices and behavior of securities are symmetric because the normal probability distribution of the Wiener process neither favours moves up nor down. But in this model we can observe a volatility smile. It is a smile with asymmetric grin up in the direction of higher draw on the credit line and negative cumulative cash flows. However, it is not due to investors getting increasingly nervous, but it is only a side effect of the contractual solution to the principal-agent problem. Importantly, in this model volatility of stock prices increases in magnitude together with the increase in leverage and credit risk.

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